LAST TIME: Fresnel diffraction with rectangular apertures, Fresnel integrals, Cornu Spiral, and introduced Fourier optics

Reminder

One of Fourier's theorems states that a perdiodic function may be written as a series of sines and cosines given by

$$f(x) = \frac{A_o}{2} + \sum_{m=1}^{\infty} A_m \cos mkx \ + \sum_{m=1}^{\infty} B_m \sin mkx \, .$$

Obviously, this expansion would not do us any good unless we have a way to determine the coefficients A_m and B_m. Fortunately, sines, cosines, and complex exponentials fall into a category of functions that are known as complete sets of orthogonal functions. As such, they have conditions that are called orthogonality conditions that allow us to determine A_m and B_m. These conditions are not hard to prove for sines and cosines, but in the interest of time, I will state what they are.

$$\int_0^{\lambda} \sin akx \cos bkx \, dx = 0\,; \int_0^{\lambda} \cos akx \cos bkx \, dx = \frac{\lambda}{2}\delta_{ab}; \int_0^{\lambda} \sin akx \sin bkx \, dx = \frac{\lambda}{2}\delta_{ab}$$

δ_{ab} is the Kronecker delta, which is 1 if $a = b$ and 0 if $a \neq b$. Many other functions have orthogonality conditions that allow for other types of expansions. If we wish to calculate the value for A_m, we just multiply our representation of the function by $\cos lkx$ and integrate over the period. Only values for $l = m$ contribute so we find

$$\int_0^{\lambda} f(x) \cos mkx \, dx = \int_0^{\lambda} A_m \cos^2 mkx \, dx = \frac{\lambda}{2} A_m$$

so

$$A_m = \frac{2}{\lambda}\int_0^{\lambda} f(x) \cos mkx \, dx.$$

We do the same multiplication by the sine function to obtain a value for B_m given by

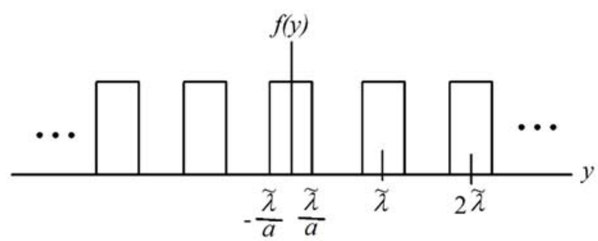
$$B_m = \frac{2}{\lambda}\int_0^{\lambda} f(x) \sin mkx \, dx.$$

Let's convert these generic equations into equations that are specific for our case in optics. To do this we make the following substitutions. Let $\lambda \to \tilde{\lambda}, x \to y,$ and $k \to \tilde{k}_Y$. Therefore, we may write the following equations given by

$$f(y) = \frac{A_o}{2} + \sum_{m=1}^{\infty} A_m \cos m\tilde{k}_Y y \ + \sum_{m=1}^{\infty} B_m \sin m\tilde{k}_Y y \, ,$$

$$\int_0^{\tilde{\lambda}} \sin a\tilde{k}_Y y \cos b\tilde{k}_Y y \, dy = 0\,; \int_0^{\tilde{\lambda}} \cos a\tilde{k}_Y y \cos b\tilde{k}_Y y \, dy = \frac{\tilde{\lambda}}{2}\delta_{ab}; \int_0^{\tilde{\lambda}} \sin a\tilde{k}_Y y \sin b\tilde{k}_Y y \, dy = \frac{\tilde{\lambda}}{2}\delta_{ab}, \;\; A_m = \frac{2}{\tilde{\lambda}}\int_0^{\tilde{\lambda}} f(y)\cos m\tilde{k}_Y y \, dy, \;\; \textit{and } B_m = \frac{2}{\tilde{\lambda}}\int_0^{\tilde{\lambda}} f(y)\sin m\tilde{k}_Y y \, dy.$$

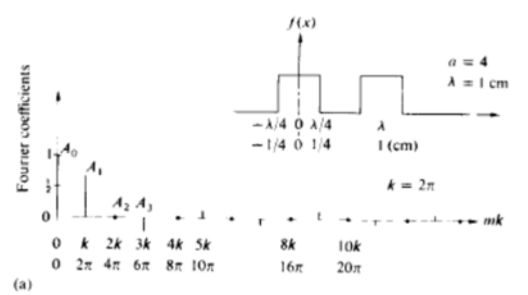
It is useful to pay attention to whether the function is even or odd because, since the sine is odd $f(y) = -f(-y)$ and the cosine is even $f(y) = f(-y)$, $B_m = 0$ for even functions and $A_m = 0$ for odd functions. These follow from the orthogonality conditions. Consider a function that would simulate the Ronchi ruling we mentioned and saw demonstrated in the video. Its aperture function would be represented by the following graph.

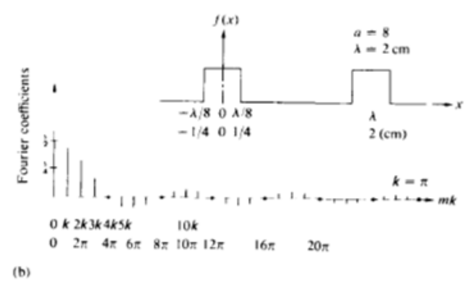


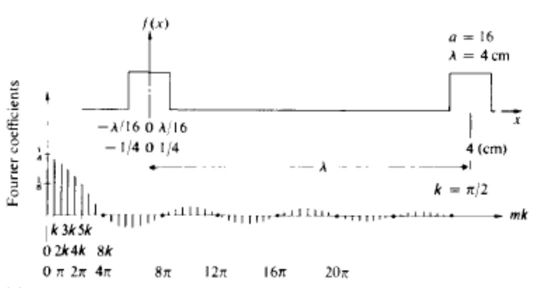
The width of the peaks is given by $2\frac{\tilde{\lambda}}{a}$ with a constant period of $\tilde{\lambda}$. We may vary the peak width and period independently of one another. The Fourier series is not hard to work out by doing the integrals, and the result is given by

$$f(y) = \frac{2}{a} + \sum_{m=1}^{\infty} \frac{4}{a}\operatorname{sinc}\left(\frac{2m\pi}{a}\right)\cos m\,\tilde{k}_Y y,$$

Where you will recall the the sinc function is just given by (sin x)/x. We can imagine getting to a single pulse instead of a periodic set of pulses by stretching out the period and narrowing the pulse. Here are some figures from Hecht that shows how the process works.







As this process continues, we reach a point where the spacing between the values of As is continuous, and the sums we used are replaced by integrals. This means that

$$f(y) = \frac{1}{\pi}\left[\int_0^{\infty} A(\tilde{k}_Y)\cos(\tilde{k}_Y y)d\tilde{k}_Y + \int_0^{\infty} B(\tilde{k}_Y)\sin(\tilde{k}_Y y)d\tilde{k}_Y\right]$$

where

$$A(\tilde{k}_Y) = \int_{-\infty}^{+\infty} f(y)\cos(\tilde{k}_Y\, y) \text{ and } B(\tilde{k}_Y) = \int_{-\infty}^{+\infty} f(y)\sin(\tilde{k}_Y\, y)\, dy$$

Because we know how the sine, cosine, and complex exponential functions are related, it should not be a surprise to you that we can combine everyting into two integrals given generically by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(k)e^{-ikx}\, dk \text{ and } F(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}\, dx$$

In our notation specifically written for optics, we obtain

$$\mathcal{A}(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(\tilde{k}_Y)e^{-i\tilde{k}_Y y}\, d\tilde{k}_Y \text{ and } E(\tilde{k}_Y) = \int_{-\infty}^{+\infty} \mathcal{A}(y)e^{i\tilde{k}_Y y}\, dy.$$

This establishes the connection between the aperture function $\mathcal{A}(y)$ and $E(\tilde{k}_Y)$. The aperture function and the electric field are Fourier transform pairs. Physically, what these equations say is that the electric field at a point on the screen is made up of a superposition of plane wave of a certain amplitude and phase that propagate from the aperture to the observation point. Likewise, from the reversibility of light, the aperture function may be considered to be made up from a superposition of plane waves that propagate from points on the observation plane back to the aperture. Notice the difference in signs of the exponent in the two integrals. Our one-dimensional diffraction pattern is nothing more than this superposition of plane waves, a point that we have already made numerous times.

To draw an analogy between the time-frequency domains and the space-spatial frequency domains, recall that you might have first seen the Fourier transform written as

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{i\omega t}\, dt \text{ and } f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{-i\omega t}\, d\omega.$$

Consider the following aperture function given by

$$\mathcal{A}(y) = 1 + \cos(\tilde{k}_o\, y)$$

so

$$E(\tilde{k}_Y) = \int_{-\infty}^{+\infty} [1 + \cos(\tilde{k}_o\, y)]\, e^{i\tilde{k}_Y y} dy$$

$$= \int_{-\infty}^{-\infty} e^{i(\tilde{k}_Y - 0)y}\, dy + \frac{1}{2}\int_{-\infty}^{-\infty} e^{i(\tilde{k}_Y + \tilde{k}_o)y} dy + \frac{1}{2}\int_{-\infty}^{-\infty} e^{i(\tilde{k}_Y - \tilde{k}_o)y}\, dy.$$

Note that this aperture function is an infinite in extent and is a raised cosine function. To handle these types of integrals, we need to introduce the Dirac delta function, a limit of a sequence of functions, that is made to deal with singularities produced by points.

Dirca delta functions:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 1 \end{cases}$$

and

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1$$

One of the most important properties is its sifting property given by

$$\int_{-\infty}^{+\infty} \delta(x)f(x)dx = f(0) \quad \text{or} \quad \int_{-\infty}^{+\infty} \delta(x - x_o)f(x)dx = f(x_o)$$

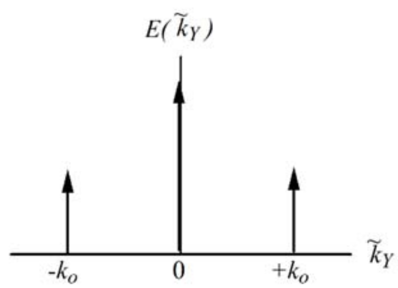
Dirac delta functions have the following property.

$$\delta(k - k') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i(k-k')x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(k-k')x}\,dx$$

Now we use this property to evaluate our integrals to obtain

$$E(\tilde{k}_Y) = 2\pi\delta(0) + \pi\delta(\tilde{k}_Y - -\tilde{k}_o) + \pi\delta(\tilde{k}_Y - \tilde{k}_o).$$

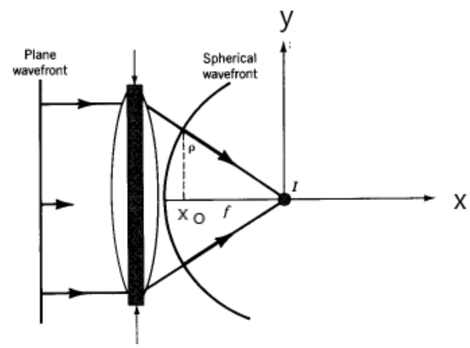
The diffraction pattern (Fourier transform) of an infinite raised cosine function only has 3 spatial frequency components. We draw $E(\tilde{k}_Y)$ as



As you might expect, since the function is a pure cosine wave shifted upward by 1, it requires only the three components to make up the aperture. Notice that this is not the same as a truncated cosine function that might extend from -*b*/2 to +*b*/2. We would have to do the integrals over much different values and delta functions would not be involved. Can you guess what this might look like?

Now let's look at how a lens behaves in the wave theory of light. In the geometric optics formalism, a lens simply refracted light and we could determine the connections between the object, image, and focal length by involing the object-image matrix.

Here is a figure that shows the essential features of a lens and what it does in the wave theory model. We know that the collimated light (plane wave) changes into a spherical (converging) beam as it produces a point in geometric optics. Now, we know that the finite size of the lens causes diffraction, so the image is a diffraction pattern. This process works because the center of the lens is thicker and has a larger optical path length than at the edges. We now know that optical path length is connected to the phase change given by



$$\frac{\phi}{2\pi} = \frac{OPD}{\lambda}.$$

From our figure, we obtain

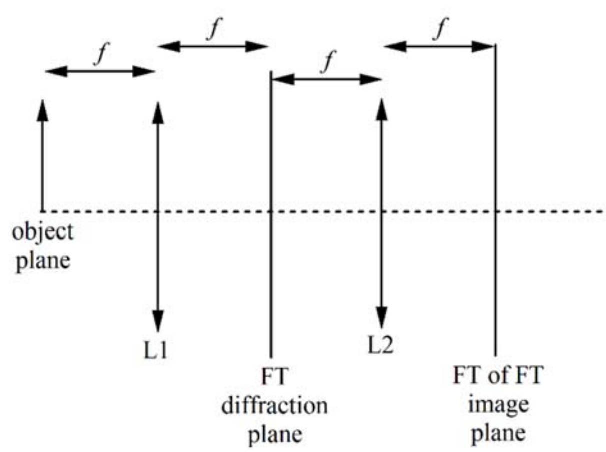
$$(f - x_o)^2 + \rho^2 = f^2,$$

so

$$f^2 - 2fx_o + x_o^2 + \rho^2 = f^2 \Longrightarrow 2fx_o = \rho^2 - x_o^2,$$

but we usually limit our imaging to paraxial conditions where $x_o^2 \ll 1$ so $x_o \cong \frac{\rho^2}{2f}$. Therefore, the phase change introduced by the lens in the paraxial approximation is given by $\phi = kx_o$. If we now look at a spherical wave incident on the lens, we see exactly how the lens delays the center part of the wave and allows the part at the edges to "catch up" to the center part and form an outgoing plane (collimated) wave.

Let's look at an optical system that is known as a 4-f system. Here is a figure that shows how the system works. Let's use our raised cosine aperture function as the object so we can follow what happens. We know that there is one bright spot in the middle and two spots that are diminished in size equidistant from it. What is the Fourier transform of a Fourier transform? Sometimes, the Fourier transform is abbreviated as script F, so we might write our expression on the last plane as $\mathfrak{F}\{\mathfrak{F}[f(y)]\}$. It turns out that the this expression can be shown to be given by



$$\mathfrak{F}\{\mathfrak{F}[f(y)]\} = 2\pi f(-y).$$

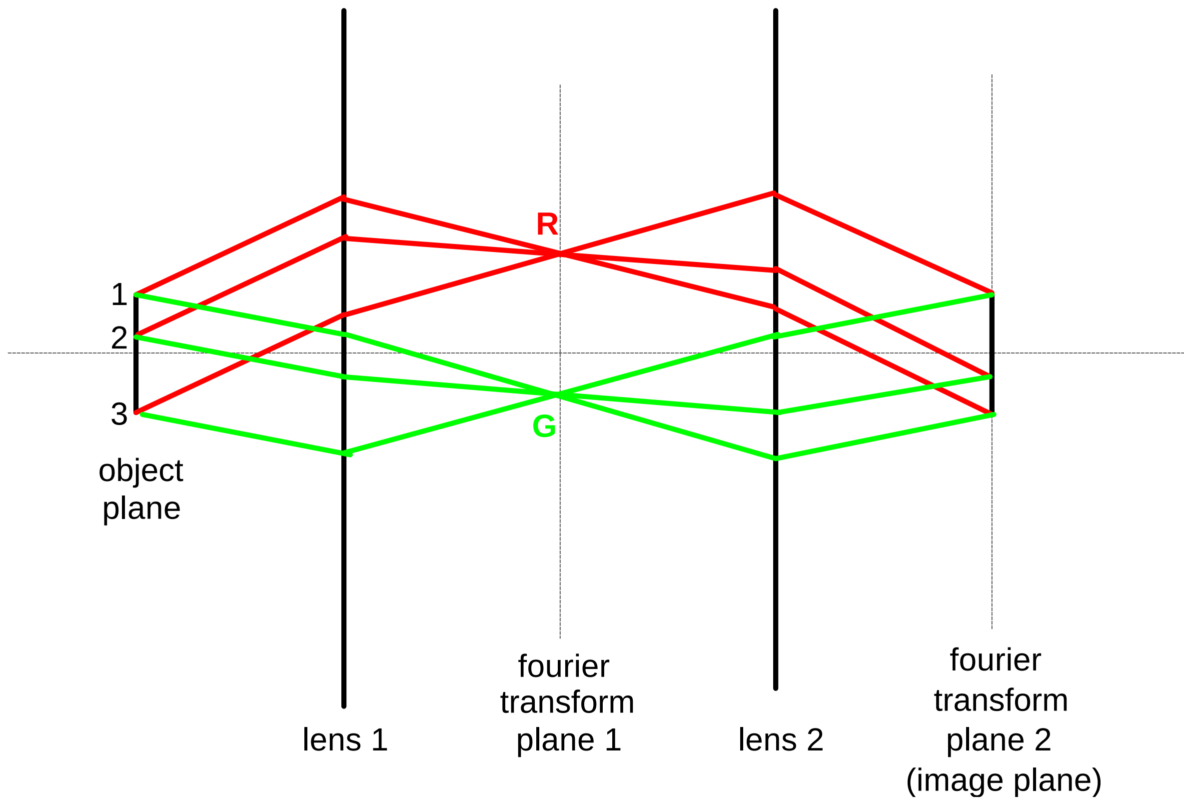
This means that on the FT of FT plane is the image. That is, the function f is just inverted and multiplied by a constant. We know that our image should be the same size as our object, so we may ignore the constant. How do we manipulate the diffraction plane to change the image. Now that we have discrete points on the diffraction plane, it is easy to block (filter) out one or more of the spots. This is called spatial frequency filtering and has many very nice uses. First, consider what happens if we block out the two weaker signals. Then,we have only one central point lying in the fron focal plane of a lens. What comes out from the first lens? From geometric optics, we would say that we get a collimated beam of light. In wave optics, we would get the FT of the of the delta function $\delta(\tilde{k}_Y - 0)$, given by

$$\int_{-\infty}^{+\infty} \delta(\tilde{k}_Y - 0) e^{i\tilde{k}_Y y} \, d\tilde{k}_Y = e^{i(0)y} = 1.$$

What do we expect if we filter out the central spot? Now we have two point sources, and we should obtain Young's interference. In equation form,

$$e^{i\tilde{k}_o y} + e^{-i\tilde{k}_o y} = 2\cos(\tilde{k}_o y).$$

The figure and explanation on the following page shows an interesting way to look at the full picture to connect geometrical optics to wave optics. Explain how a single lens produces an image.



The drawing above illustrates the connection between the geometrical interpretation of image formation and the wave (Fourier transform) interpretation of image formation. The object lies in the front focal plane of lens 1. The lenses have the same focal length and are separated by twice the focal length so that the final image is formed in the back focal plane of lens 2. The drawing shows three points on the object plane, each with two **k**-vectors (rays) emanating from it. The **k**-vectors in red are all parallel, and they focus at the point R on the Fourier transform (diffraction) plane after passing through lens 1. Three additional **k**-vectors (shown in green) are emitted in a different direction from the same three points; these vectors focus at the point G after passing through lens1. The points R and G represent the strengths of two different **k**-vectors that make up the object. Each provides partial information about the object, but this information comes from the entire object. Fourier transform plane 1 serves as the object plane for lens 2, so Fourier transform plane 2 is the Fourier transform of the Fourier transform of the object. Symbolically, this gives $\mathcal{F}\{\mathcal{F}\{f(x)\}\}$, where $\mathcal{F}$ denotes the Fourier transform, and f(x) is the function that represents the object. Now, there are three sets of parallel **k**-vectors instead of two as before, two from each of the three points on the original object. The drawing is scaled properly so that these vectors are obvious. Each set of parallel **k**-vectors focuses on Fourier transform plane 2 and gives the inverted image with the same size as the object here because lens 1 and lens 2 have the same focal length. Notice that each set of two parallel **k**-vectors incident on lens 2 comes from *different* points on Fourier transform plane 1, but each set comes from the *same* point on the object. This mixing of incoming parallel **k**-vectors is what makes the image; it also explains why the image is the Fourier transform of a Fourier transform and not the inverse Fourier transform of a Fourier transform.