Group Theory Review

1 Groups, Rings and Fields

Definition 1. *A group consists of a set G and a binary operation "$\cdot$" defined on G, for which the following conditions are satisfied:*

1. *Associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in G$.*
2. *Identity: There exists an element $1 \in G$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in G$.*
3. *Inverse: Given $a \in G$, there exists $b \in G$ such that $a \cdot b = b \cdot a = 1$.*

Examples: $\mathbb{S}_n$ of permutations on $[n]$, $GL(n, \mathbb{R})$ of real nonsingular $n \times n$ matrices under matrix multiplication.

If we drop the condition on the existence of an inverse, we obtain a **monoid**. Note that a monoid always has at least one element, the identity. As an example, given a set S, then the set of all strings of elements of S is a monoid, where the monoid operation is string concatenation and the identity is the empty string λ. Monoids are also known as semigroups with identity.

Definition 2. *A ring consists of a set R and two binary operations "+" (addition) and "$\cdot$" (multiplication), defined on R, for which the following conditions are satisfied:*

1. *Additive associative: $(a + b) + c = a + (b + c)$, for all $a, b, c \in R$.*
2. *Additive commutative: $a + b = b + a$, for all $a, b \in R$.*
3. *Additive identity: There exists an element $0 \in R$ such that for all $a \in R$, $0 + a = a + 0 = a$.*
4. *Additive inverse: for every $a \in R$, there exists $-a \in R$ such that $a + (-a) = (-a) + a = 0$.*
5. *Left and right distributivity: For all $a, b, c \in R$, it holds that $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.*
6. *Multiplicative associativity: For all $a, b, c \in R$, it holds that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.*

If R is multiplicatively commutative, then it is a commutative ring.
Examples: $\mathbb{Z}$ under the usual addition and multiplication, $\mathbb{R}[x], \mathbb{C}[x]$.

Definition 3. *A field consists of a set F and two binary operations "+" (addition) and "·" (multiplication), defined on R, for which the following conditions are satisfied:*

1. *$(F, +, \cdot)$ is a ring.*
2. *Multiplicative commutative: For any $a, b \in F$, $a \cdot b = b \cdot a$.*
3. *Multiplicative identity: There exists $1 \in F$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in F$.*
4. *Multiplicative inverse: If $a \in F$ and $a \neq 0$, there exists $b \in F$ such that $a \cdot b = b \cdot a = 1$.*

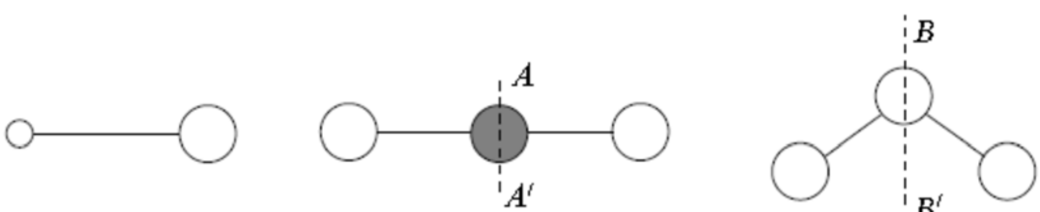
Examples: $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}(x), \mathbb{R}(x), \mathbb{C}(x)$.

A field k is algebraically closed if every nonzero polynomial $p(x) \in k[x_1, \ldots, x_n]$ has at least a root in k.

We looked at several different multiplication (operation) tables and found that many concrete representations of groups are actually the same group. The triangle and the permutation group have the same operation table, so they are really the same group – just different manifestations of the same thing.

What is the main idea we want to get out of group theory and its applications to physics and the physical sciences in general? If we have a particular representation of a group, what can we predict about a physical situation? A couple of examples might help you understand this better.

Suppose we are interested in knowing by the symmetry and hence the group that describes this symmetry. Does a particular molecule or crystal class have a permanent electric or magnetic dipole? A second consideration occurs in optics when we would like to know if a certain response to an optical or electric field occurs. The first of these is simpler, so we look at it first.



(a) HCl (b) CO_2 (c) O_3

The three molecules shown all have different symmetries. If any charge distribution has the center of positive charge and the center of negative charge coinciding, then no permanent electric dipole should exist. For HCl, we presume that the negative charge on the H atom shifts because of the Cl atom, so it can have a permanent electric dipole. It seems obvious that the linear CO2 molecule cannot have a permanent dipole moment. In terms of group theory and symmetry operations, we note that carries the molecule into itself, so there can be no dipole perpendicular to that direction. Rotation about AA' or reflection carries the molecule into itself, so it cannot have a dipole moment there either. For the ozone molecule, no symmetry operation perpendicular to BB' carries the molecule into itself, so a dipole parallel to BB' is possible. For the direction perpendicular to BB', however, either a reflection or pi rotation carries the molecule into itself so no dipole perpendicular to BB' is possible.

In optics, electric fields may be applied to change the characteristics of a material. The two most common effects are the linear, Pockels effect and the quadratic, Kerr effect. The existence of these depends on the point group of the particular crystal being studied. These effects are characterized by the equation given by

$$P_i = \epsilon_o \chi_{ij}^{(1)} E_j + \epsilon_o \chi_{ijk}^{(2)} E_j E_k + \epsilon_o \chi_{ijkl}^{(3)} E_j E_k E_l + \cdots.$$

If a crystal has no inversion center, both the linear and quadratic effects are present with the linear effect typically being much larger. Usually, under these circumstances, the quadratic effect cannot be measured. Several years ago, one of my students realized that with the appropriately directed electric field, it was possible to cancel the linear effect and measure the quadratic effect. His realization came from a careful study of the symmetries involved with electric fields applied in different directions. Our measurements were the first ever under these conditions. We also realized that because of the crystal symmetry, other types of effects were also able to be measured in a similar way. Of the 32 crystal point groups, it is possible to determine from the point group which effects are present in different materials. Here is a very broad statement about symmetries.

If a physical system is such that the application of particular symmetry operations leaves the system indistinguishable from the original system, then the functions that describe its behavior must have the corresponding property of invariance when subjected to the same operations.

This is essentially the real value of group theory as applied to physical systems.

There are more ideas involved than what we have mentioned, so I will go through a few other ideas.

Classes

If *a*, *b*, and *c* are three elements of a groups and

$$b = cac^{-1},$$

then *b* is conjugate to *a*. If *b* is conjugate to *a*, then *a* is conjugate to *b*. If *a* is conjugate to *b* and *c*, then *b* and *c* are conjugate to each other. The complete set of all elements that are conjugate to each other is the class of the group. For the permutation group that we dealt with earlier, here is the class of elements conjugate to G_{12}. To find them, you just have to work out gag^{-1} for each element. I will show the operation table for the group and the table showing the class of elements conjugate to G_{12} on the following page.

	1	G_{12}	G_{13}	G_{23}	G_{132}	G_{312}
1	1	G_{12}	G_{13}	G_{23}	G_{132}	G_{312}
G_{12}	G_{12}	1	G_{132}	G_{312}	G_{13}	G_{23}
G_{13}	G_{13}	G_{312}	1	G_{132}	G_{23}	G_{12}
G_{23}	G_{23}	G_{132}	G_{312}	1	G_{12}	G_{13}
G_{132}	G_{132}	G_{23}	G_{12}	G_{13}	G_{312}	1
G_{312}	G_{312}	G_{13}	G_{23}	G_{12}	1	G_{132}

Note that this group does not commute and that is has six elements.

$G_{12}\ G_{12}\ G_{12}^{-1} = G_{12}$
$G_{13}\ G_{12}\ G_{13}^{-1} = G_{23}$
$G_{23}\ G_{12}\ G_{23}^{-1} = G_{13}$
$G_{132}\ G_{12}\ G_{132}^{-1} = G_{13}$
$G_{312}\ G_{12}\ G_{312}^{-1} = G_{23}$

So the members of this class are G_{12}, G_{13}, and G_{23}. Note that these elements simply interchange two elements, so this makes some sense. If you were to continue this process, you would find that the elements G_{312} and G_{132} also form another class, and the identity element also forms another class. The class tends to organize the elements into the type of operation they perform. Three classes exist for this group, and the sum of the disjoint classes make up the group.

Here is the multiplication table for the group 1, -1, *i*, and -*i*. Notice that this group is Abelian and has only 4 elements.

	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

If we look at the group of three elements, here is its multiplication table. Notice that it is also Abelian. You might remember that SO(2) is Abelian, but that SO(3) is not.

	1	a	b
1	1	a	b
a	a	b	1
b	b	1	a

While I will not prove this, the first nonabelian group has 6 elements. That is one reason the triangle group and permutation group are important. A second reason is that they happen to form a group that the ammonia molecule falls into, so it is a very useful group for illustration purposes.

Subgroups

Notice that for our permutation group, the set [G_{12}, 1] forms a subgroup because each element is in a group. You can see this from the multiplication table. Can we find any subgroups in the 3- or 4- element group?

Representations

A representation of the group is a g–element sequence of the square matrices (of the same dimension, each element of the group is associated to a matrix), such that the matrices have the multiplication table consistent with the multiplication table of the group. Let's consider a representation of the 3 – element group we looked at before. Here, the matrix represents the multiplication properties of a particular element. We need vectors to operate on, so they are created from the elements of the group. Let

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \quad M_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

M_1 represents 1, M_2 represents a, and M_3 represents b.

$$M_2 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}.$$

There are many other representations of the group, the complex numbers that represent rotations, two-D rotation matrices, and 3-D rotations matrices as well.

Character

The character of a representation is the set of traces of the matrices that represent the group elements. The character of the C_3 group is the set {3,0,0}.

Irreps

A representation is called reducible if its matrices can be transformed into the so-called block form by using the transformation $\mathrm{P}^{-1}\Gamma(\hat{R}_i)\mathrm{P}$ for every matrix $\Gamma(\hat{R}_i)$ whereP is a non-singular matrix.