LAST TIME: Fresnel diffraction with rectangular apertures, Fresnel integrals, and Cornu Spiral

Reminder

$$u = y\left[\frac{2(\rho_o + r_o)}{\rho_o r_o \lambda}\right]^{1/2} \text{ and } v = z\left[\frac{2(\rho_o + r_o)}{\rho_o r_o \lambda}\right]^{1/2}$$

to obtain

$$E_P = \frac{\mathcal{E}_o}{2(\rho_o + r_o)} e^{ik(\rho_o + r_o)} \int_{u_1}^{u_2} e^{i\pi u^2/2}\, du \int_{v_1}^{v_2} e^{i\pi v^2/2}\, dv.$$

These integrals over *u* and *v* are known as the Fresnel integrals. Now we define the following functions as

$$\mathcal{C}(w) = \int_0^w \cos\left(\frac{\pi w'^2}{2}\right) dw' \text{ and } \mathcal{S}(w) = \int_0^w \sin\left(\frac{\pi w'^2}{2}\right) dw'$$

so that

$$\int_0^w e^{i\pi w'^2/2}\, dw' = \mathcal{C}(w) + i\,\mathcal{S}(w)$$

Now we may expand all this to obtain

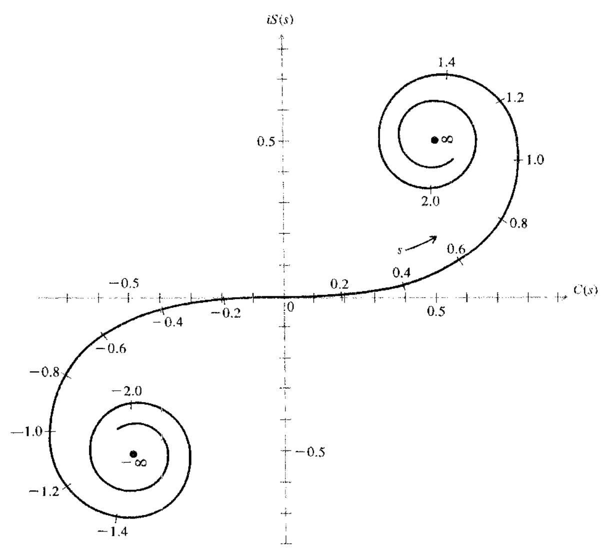
$$E_P = \frac{E_u}{2}[\mathcal{C}(u) + i\,\mathcal{S}(u)]_{u_1}^{u_2}[\mathcal{C}(v) + i\,\mathcal{S}(v)]_{v_1}^{v_2}.$$

Carrying out the algebra

$$E_P = \frac{E_u}{2}\{[\mathcal{C}(u_2) + i\,\mathcal{S}(u_2) - \mathcal{C}(u_1) - i\,\mathcal{S}(u_1)] \times [\mathcal{C}(v_2) + i\,\mathcal{S}(v_2) - \mathcal{C}(v_1) - i\,\mathcal{S}(v_1)]\}$$

Therefore,

$$I_P = \frac{I_o}{4}\{[\mathcal{C}(u_2) - \mathcal{C}(u_1)]^2 + [\mathcal{S}(u_2) - \mathcal{S}(u_1)]^2\} \times \{[\mathcal{C}(v_2) - \mathcal{C}(v_1)]^2 + [\mathcal{S}(v_2) - \mathcal{S}(v_1)]^2\}.$$



1. $ds = [dX^2 + dY^2]^{(1/2)} = A(u)\, du$ = differential arc length

2. Slope = $dY/dX = \tan[\phi(u)]$; independent of form of $\phi(u)$

3. Curvature = $K = d\phi/ds = [A(u)]^{-1} d\phi/du$

4. $$Z_{21} = Z_2 - Z_1 = \int_{u_1}^{u_2} A(u')e^{i\phi(u')}du' = \int_0^{u_2} A(u')e^{i\phi(u')}du' - \int_0^{u_1} A(u')e^{i\phi(u')}du'$$

5. A graph of $Y(u)$ versus $X(u)$ traces a vibration curve with $|Z_{21}|^2$ giving the square of the chord length between (X_1, Y_1) and (X_2, Y_2) which, in turn is proportional to the intensity of the diffraction pattern. u is a parameter that is the rescaled arc length.

One of the most important points to understand from this development is that the arc length along the spiral is usually constant. As we move our observation point from the center of the observation plane, the arc length remains constant, but the values of u_1 and u_2 change as does the chord length joining the ends of the arc. How do you think we can use the Cornu spiral to obtain Franuhofer diffraction?

Fourier Optics

Fourier optics is the further study of optics using the formality of Fourier series and the Fourier transform. Recall that we wrote the diffraction integral as

$$E(Y) = C\int \mathcal{A}(y)\, e^{iky\sin\theta} dy \text{ with } \sin\theta = \frac{Y}{R}.$$

Therefore,

$$E(Y) = C\int \mathcal{A}(y)\, e^{iky\frac{Y}{R}} dy.$$

We define

$$\tilde{k}_Y = \frac{kY}{R} = \text{angular spatial frequency} - \text{note dimensions of inverse length}$$

We now write

$$E(Y) = C\int \mathcal{A}(y)\, e^{i\tilde{k}_Y y} dy \text{ and call } E(Y) \text{ and } \tilde{k}_Y \text{ Fourier transform pairs.}$$

An important distinction is that k and λ refer to light and $\tilde{k}$ and $\tilde{\lambda}$ refer to angular spatial frequency and spatial periods, respectively. Here is an example that shows how these ideas work. We could have a periodic set of pickett fences spaced 10 cm apart so that $\tilde{\lambda}$ =10 cm, $\tilde{k} = \frac{2\pi}{\tilde{\lambda}} = 0.63\ \text{cm}^{-1}$.

$$\tilde{\kappa} = \frac{1}{\tilde{\lambda}} = 0.1\ \text{cm}^{-1}.$$

One of Fourier's theorems states that a perdiodic function may be written as a series of sines and cosines given by

$$f(x) = \frac{A_o}{2} + \sum_{m=1}^{\infty} A_m \cos mkx \; + \; \sum_{m=1}^{\infty} B_m \sin mkx \, .$$

Obviously, this expansion would not do us any good unless we have a way to determine the coefficients A_m and B_m. Fortunately, sines, cosines, and complex exponentials fall into a category of functions that are known as complete sets of orthogonal functions. As such, they have conditions that are called orthogonality conditions that allow us to determine A_m and B_m. These conditions are not hard to prove for sines and cosines, but in the interest of time, I will state what they are.

$$\int_0^{\lambda} \sin akx \cos bkx \, dx = 0\,; \; \int_0^{\lambda} \cos akx \cos bkx \, dx = \frac{\lambda}{2}\delta_{ab}; \; \int_0^{\lambda} \sin akx \sin bkx \, dx = \frac{\lambda}{2}\delta_{ab}$$

δ_{ab} is the Kronecker delta, which is 1 if $a = b$ and 0 if $a \neq b$. Many other functions have orthogonality conditions that allow for other types of expansions. If we wish to calculate the value for A_m, we just multiply our representation of the function by $\cos lkx$ and integrate over the period. Only values for $l = m$ contribute so we find

$$\int_0^{\lambda} f(x) \cos mkx \, dx = \int_0^{\lambda} A_m \cos^2 mkx \, dx = \frac{\lambda}{2} A_m$$

so

$$A_m = \frac{2}{\lambda}\int_0^{\lambda} f(x) \cos mkx \, dx.$$

We do the same multiplication by the sine function to obtain a value for B_m given by

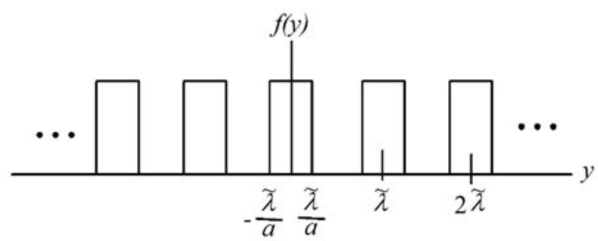
$$B_m = \frac{2}{\lambda}\int_0^{\lambda} f(x) \sin mkx \, dx.$$

Let's convert these generic equations into equations that are specific for our case in optics. To do this we make the following substitutions. Let $\lambda \to \tilde{\lambda}, x \to y,$ and $k \to \tilde{k}_Y$. Therefore, we may write the following equations given by

$$f(y) = \frac{A_o}{2} + \sum_{m=1}^{\infty} A_m \cos mk_Y y \; + \; \sum_{m=1}^{\infty} B_m \sin mk_Y y \, ,$$

$$\int_0^{\tilde{\lambda}} \sin a\tilde{k}_Y y \cos bk_Y y \, dy = 0\,; \; \int_0^{\tilde{\lambda}} \cos a\tilde{k}_Y y \cos bk_Y y \, dy = \frac{\tilde{\lambda}}{2}\delta_{ab}; \; \int_0^{\tilde{\lambda}} \sin a\tilde{k}_Y y \sin b\tilde{k}_Y y \, dy$$
$$= \frac{\tilde{\lambda}}{2}\delta_{ab}, \; A_m = \frac{2}{\tilde{\lambda}}\int_0^{\tilde{\lambda}} f(y) \cos m\tilde{k}_Y y \, dy, \; and \; B_m = \frac{2}{\tilde{\lambda}}\int_0^{\tilde{\lambda}} f(y) \sin m\tilde{k}_Y y \, dy.$$

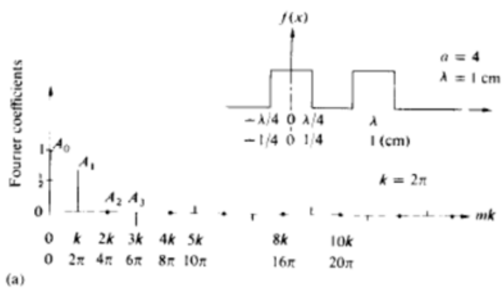
It is useful to pay attention to whether the function is even or odd because, since the sine is odd $f(y) = -f(-y)$ and the cosine is even $f(y) = f(-y)$, $B_m = 0$ for even functions and $A_m = 0$ for odd functions. These follow from the orthogonality conditions. Consider a function that would simulate the Ronchi ruling we mentioned and saw demonstrated in the video. Its aperture function would be represented by the following graph.

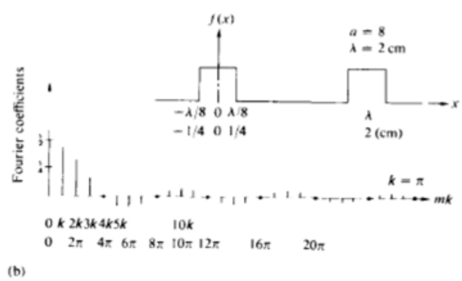


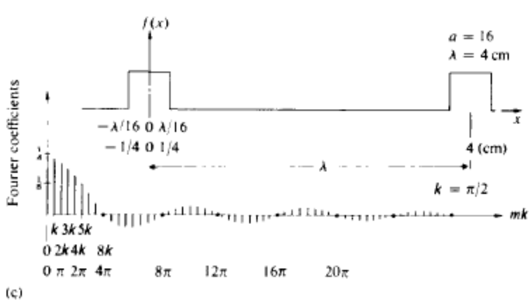
The width of the peaks is given by $2\frac{\tilde{\lambda}}{a}$ with a constant period of $\tilde{\lambda}$. We may vary the peak width and period independently of one another. The Fourier series is not hard to work out by doing the integrals, and the result is given by

$$f(y) = \frac{2}{a} + \sum_{m=1}^{\infty} \frac{4}{a} \operatorname{sinc}\left(\frac{2m\pi}{a}\right) \cos m\, \tilde{k}_Y y,$$

Where you will recall the the sinc function is just given by (sin x)/x. We can imagine getting to a single pulse instead of a periodic set of pulses by stretching out the period and narrowing the pulse. Here are some figures from Hecht that shows how the process works.







As this process continues, we reach a point where the spacing between the values of As is continuous, and the sums we used are replaced by integrals. This means that

$$f(y) = \frac{1}{\pi}\left[\int_0^{\infty} A(\tilde{k}_Y)\cos(\tilde{k}_Y y)d\tilde{k}_Y + \int_0^{\infty} B(\tilde{k}_Y)\sin(\tilde{k}_Y y)d\tilde{k}_Y\right]$$

where

$$A(\tilde{k}_Y) = \int_{-\infty}^{+\infty} f(y)\cos(\tilde{k}_Y\, y) \text{ and } B(\tilde{k}_Y) = \int_{-\infty}^{+\infty} f(y)\sin(\tilde{k}_Y\, y)\, dy$$

Because we know how the sine, cosine, and complex exponential functions are related, it should not be a surprise to you that we can combine everyting into two integrals given generically by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(k)e^{-ikx}\,dk \text{ and } F(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}\,dx$$

In our notation specifically written for optics, we obtain

$$\mathcal{A}(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E(\tilde{k}_Y)e^{-i\tilde{k}_Y y}\,d\tilde{k}_Y \text{ and } E(\tilde{k}_Y) = \int_{-\infty}^{+\infty} \mathcal{A}(y)e^{i\tilde{k}_Y y}\,dy.$$

This establishes the connection between the aperture function $\mathcal{A}(y)$ and $E(\tilde{k}_Y)$. The aperture function and the electric field are Fourier transform pairs. Physically, what these equations say is that the electric field at a point on the screen is made up of a superposition of plane wave of a certain amplitude and phase that propagate from the aperture to the observation point. Likewise, from the reversibility of light, the aperture function may be considered to be made up from a superposition of plane waves that propagate from points on the observation plane back to the aperture. Notice the difference in signs of the exponent in the two integrals. Our one-dimensional diffraction pattern is nothing more than this superposition of plane waves, a point that we have already made numerous times.

To draw an analogy between the time-frequency domains and the space-spatial frequency domains, recall that you might have first seen the Fourier transform written as

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{i\omega t}\,dt \text{ and } f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{-i\omega t}\,d\omega.$$

To illustrate the use of a different aperture function, let's look at another common aperture function. Consider an aperture function given by

$$\mathcal{A}(y) = \cos\frac{\pi y}{b} \qquad -\frac{b}{2} \le y \le \frac{b}{2}$$
$$= 0 \qquad \text{elsewhere.}$$

Now

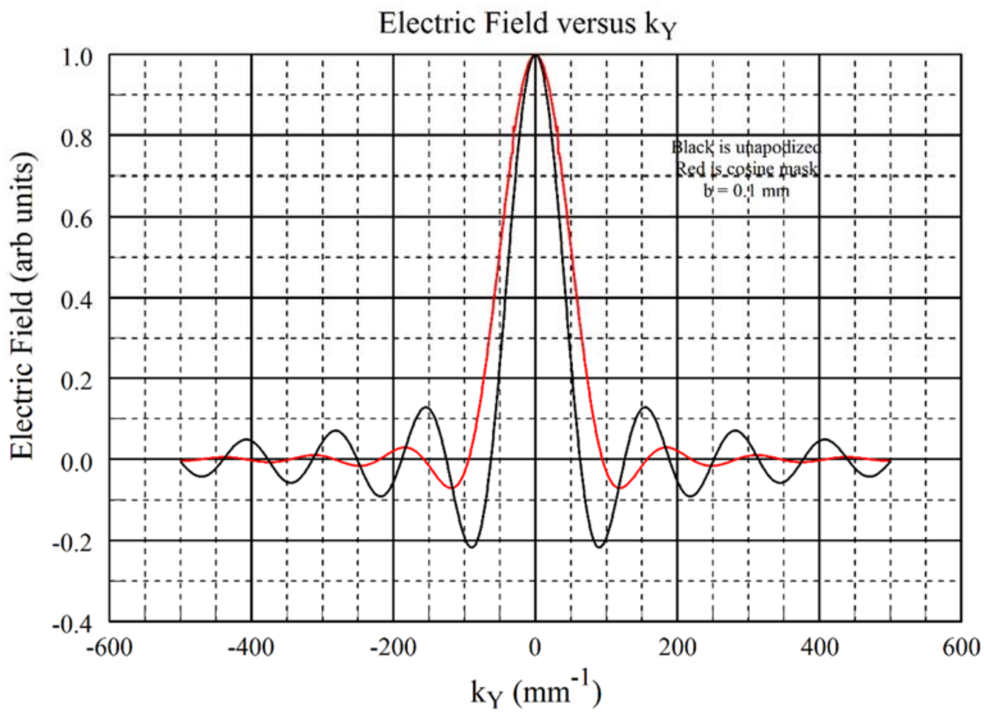
$$E(\tilde{k}_Y) = \int_{-\infty}^{+\infty} \mathcal{A}(y)e^{i\tilde{k}_Y y}\,dy$$

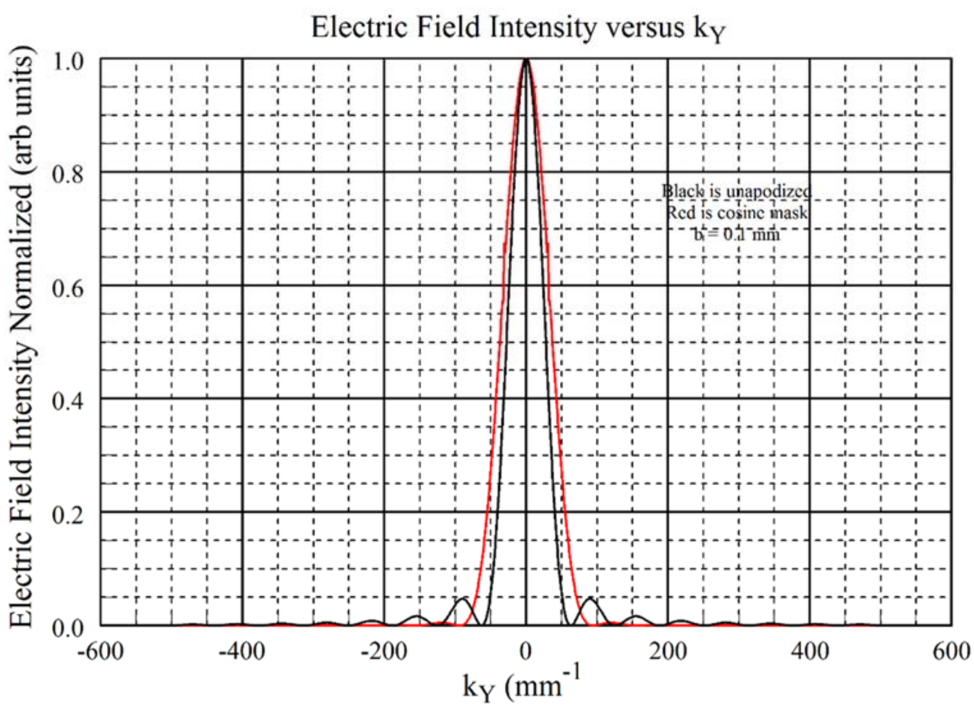
$$= \int_{-b/2}^{+b/2} \cos\frac{\pi y}{b} e^{i\tilde{k}_Y y}\,dy = \int_{-b/2}^{+b/2} \frac{1}{2}\left(e^{i\pi y/b} + e^{-i\pi y/b}\right)e^{i\tilde{k}_Y y}$$

so

$$E(\tilde{k}_Y) = \cos\left(\frac{\tilde{k}_Y b}{2}\right)\left[\frac{1}{\tilde{k}_Y + \pi/b} - \frac{1}{\tilde{k}_Y - \pi/b}\right]$$

Graphs of the electric field amplitude and electric field intensities are shown on the next page.





Next, consider the following aperture function given by

$$\mathcal{A}(y) = 1 + \cos(\tilde{k}_o\, y)$$

so

$$E(\tilde{k}_Y) = \int_{-\infty}^{+\infty} [1 + \cos(\tilde{k}_o\, y)]\, e^{i\tilde{k}_Y y} dy$$

$$= \int_{-\infty}^{-\infty} e^{i(\tilde{k}_Y - 0)y}\, dy + \frac{1}{2}\int_{-\infty}^{-\infty} e^{i(\tilde{k}_Y + \tilde{k}_o)y} dy + \frac{1}{2}\int_{-\infty}^{-\infty} e^{i(\tilde{k}_Y - \tilde{k}_o)y}\, dy.$$

Note that this aperture function is an infinite in extent and is a raised cosine function. To handle these types of integrals, we need to introduce the Dirac delta function, a limit of a sequence of functions, that is made to deal with singularities produced by points.

Dirca delta functions:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 1 \end{cases}$$

and

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1$$

One of the most important properties is its sifting property given by

$$\int_{-\infty}^{+\infty} \delta(x) f(x) dx = f(0) \quad or \quad \int_{-\infty}^{+\infty} \delta(x - x_o) f(x) dx = f(x_o)$$

NEXT TIME: Continue with Fourier optics