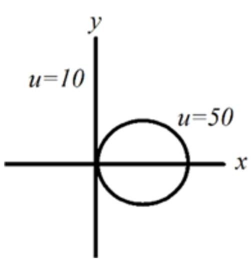
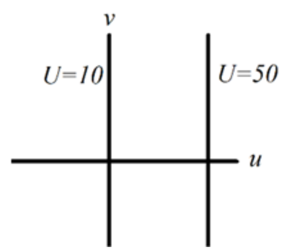
LAST TIME: Finished contour integration examples

The basic goal of conformal mapping is to figure out how to map a relatively complicated problem into a simpler one. One typical mapping occurs when a problem has mixed geometry, such as a circular hole cut into a square plate. It would seem necessary to use both Cartesian and polar coordinates to solve such a problem. Instead of try this approach, we see if we can map the circle and plate into the same geometry which is easier to solve when all the objects have the same form. We solve the problem in the simpler geometry and transform back into the original problem. The single biggest problem in this approach is determining the mapping function for a specific situation.

An example is helpful to understand the process. The example is one where steady state temperature has been established in a plate with a hole in it.




We know that Laplace's equation in the x-y plane is valid. The mapping $w = f(z) = \frac{1}{z}$ takes the unit circle into a plane at $u = 1$ and is infinite in the $v$ direction to obtain the figure shown. Notice the enormous simplification that has occurred. In the *x-y* plane, we have a mixed geometry comprised of a circle and a plane. The problem is certainly two-dimensional, and it is not at all clear how to solve Laplace's equation in that type of geometry. However, in the mapped plane, the problem is not only Cartesian like, but it is also one-dimensional, so Laplace's equation is easy to solve. We solve Laplace's equation in $U(u,v)$ and use the inverse map to get a solution in $x$ and $y$. The solution to the 1-D equation is given by

$$U(u,0) = Au + B.$$

Apply the BCs to get $U(u,0) = 10 + 40u$. To obtain the solution in the x-y plane we just need to find $u(x,y)$ from the mapping. To do that we write

$$\frac{1}{z} = u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \Longrightarrow u = \frac{x}{x^2+y^2} \text{ and}$$

finally

$$u(x,y) = 10 + 40\frac{x}{x^2+y^2}.$$

Notice one important point with this approach. Usually, solutions to Laplace's equation are expressed in terms of a sum, but here, we obtain a nice closed form solution. Because the map is one of the most important parts of the solution, let's just check to be sure that our map is correct. We found

$$u = \frac{x}{x^2+y^2},$$

So the value of $u = 1$ yields

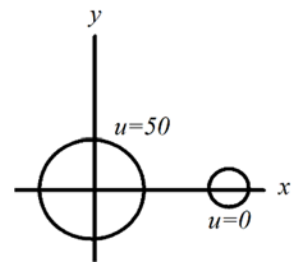
$$u = 1 = \frac{x}{x^2 + y^2} \Longrightarrow x^2 + y^2 - x = 0.$$

We complete the square to get

$$(x - 1/2)^2 + y^2 = \left(\frac{1}{2}\right)^2,$$

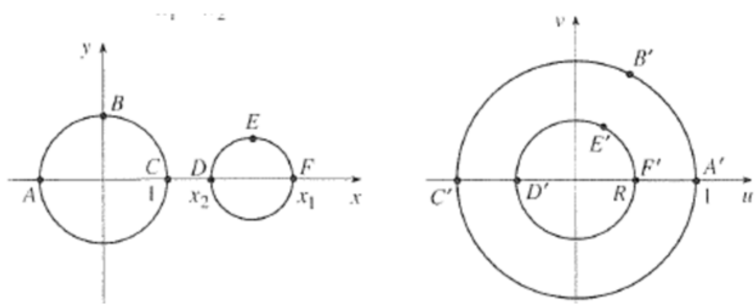
which gives us our unit circle centered at $x = \frac{1}{2}$ as required.

Let's consider the more difficult problem of determining the potential when two cylinders are held at different potentials as shown. The most general mapping we have is the bilinear map given by




$$w = \frac{az + \beta}{\gamma z + \delta}.$$

The easiest problem we can think of to solve with two cylinders is the case of concentric cylinders. To create that situation, we clearly need to translate one or both of the cylinders, but since our map acts on both objects, it will probably need to do more than that. Fortunately, others have created table of very specific maps for many different cases. Although I did not indicate sizes of the cylinders here, we need to specify the left and right intercepts of the smaller cylinder here. Here is a figure of the situation in more detail taken from the book *Advanced Mathematics for Engineers* by Michael Greenberg, although similar tabl*es* are found elsewhere. The mathematics behind the mapping is shown below. For this problem, $x_1 = 3, x_2 = 2,$ and $a = \frac{7+2\sqrt{6}}{5}$. The radius R of the smaller cylinder is given by $5 - 2\sqrt{6}$.




$$w = \frac{z - a}{az - 1}; \quad a = \frac{x_1 x_2 + 1 + \sqrt{(x_1^2 - 1)(x_2^2 - 1)}}{x_1 + x_2}, \qquad R = \frac{x_1 x_2 - 1 - \sqrt{(x_1^2 - 1)(x_2^2 - 1)}}{x_1 - x_2}, \quad 1 < x_2 < x_1$$

Now, we are in a position to use polar coordinates for Laplace's equation with 2 concentric cylinders, a problem we have solve many times. Because no $\phi -$ dependence occurs, the equation is rather simple and is given by

$$\frac{d^2U}{d\rho^2} + \frac{1}{\rho}\frac{dU}{d\rho} = 0.$$

The solution to this ODE is given by

$$U(\rho) = A + B \ln \rho.$$

We use the boundary conditions $U(R) = 0$ and $U(1) = 50$ to obtain

$$U(\rho) = 50\left(1 - \frac{\ln \rho}{\ln R}\right).$$

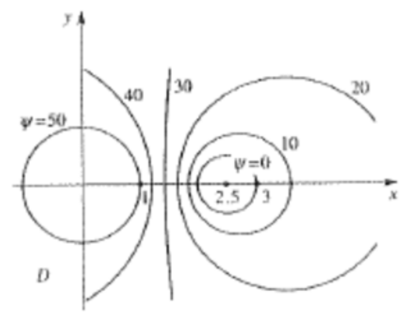
We convert back to the x-y coordinates in the z-plane to obtain

$$\rho = \sqrt{ww^*} = \sqrt{\frac{(x-a)+iy}{(ax-1)+iay}\frac{(x-a)-iy}{(ax-1)-iay}} = \sqrt{\frac{(x-a)^2+y^2}{(ax-1)^2+a^2y^2}}$$

and

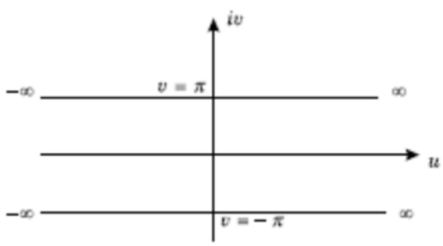
$$u(x,y) = \frac{50}{\ln R}\left[\ln R - \frac{1}{2}\ln\frac{(x-a)^2+y^2}{(ax-1)^2+a^2y^2}\right].$$

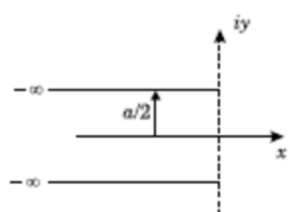
A few of the equipotential surfaces are shown in the figure to the left. Because we know that the electric field lines are perpendicular to the constant potential lines, we can sketch them in if we wish to do so. The function $v(x,y)$ also gives the electric field function. Another interesting problem concerns the fringing fields at the edge of a parallel plate capacitor. The map that takes us from a parallel-plate capacitor of infinite length to semi-infinite plates is given by




$$z = \frac{a}{2\pi}(1 + w + e^w)$$

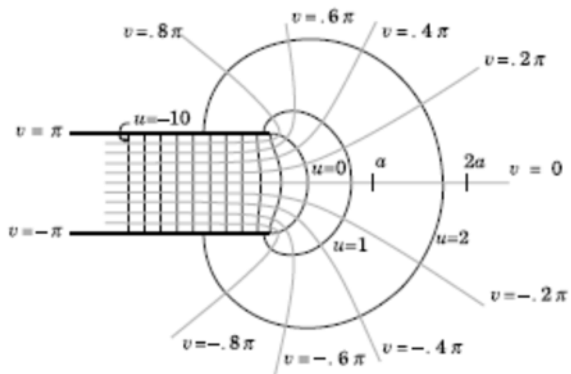
This time, we consider the parallel-plate capacitor in the $u - v$ plane and get the fringing fields in the $z$-plane. Here is the mapping




The algebra is messy, but here are the potentials and fields determined from such a mapping.

$$x = \frac{a}{2\pi}(1 + u + e^u \cos v) \text{ and } y = \frac{a}{2\pi}(v + e^u \sin v).$$




How are these mappings determined? Give a short account. Here are a few of the many mappings that are usually found in tables

NEXT TIME: Solutions to general partial differential equations without using separation of variables.