LAST TIME: Fresnel diffraction with rectangular apertures, Fresnel integrals, and Cornu Spiral

REMINDER: We started with the diffraction integral without the obliquity factor given by

$$E_P = \int_{y_1}^{y_2} \int_{z_1}^{z_2} \mathcal{E}_o \frac{e^{i[k(\rho + r)]}}{\rho r \lambda} dS,$$

Making the same approximations as before yielded

$$E_P = \frac{\mathcal{E}_o}{\rho_o r_o \lambda} \int_{y_1}^{y_2} \int_{z_1}^{z_2} e^{i\left[k\left(\rho_o + r_o + (y^2+z^2)\left(\frac{\rho_o + r_o}{2\rho_o r_o}\right)\right)\right]} dzdy.$$

Simplifying gives

$$E_P = \frac{\mathcal{E}_o}{\rho_o r_o \lambda} e^{ik(\rho_o + r_o)} \int_{y_1}^{y_2} \int_{z_1}^{z_2} e^{i\left[k\left((y^2+z^2)\left(\frac{\rho_o + r_o}{2\rho_o r_o}\right)\right)\right]} dzdy.$$

Now we make some substitutions that make this integral look a bit more standard. We let

$$u = y\left[\frac{2(\rho_o + r_o)}{\rho_o r_o \lambda}\right]^{1/2} \text{ and } v = z\left[\frac{2(\rho_o + r_o)}{\rho_o r_o \lambda}\right]^{1/2}$$

to obtain

$$E_P = \frac{\mathcal{E}_o}{2(\rho_o + r_o)} e^{ik(\rho_o + r_o)} \int_{u_1}^{u_2} e^{i\pi u^2/2} du \int_{v_1}^{v_2} e^{i\pi v^2/2} dv.$$

These integrals over *u* and *v* are known as the Fresnel integrals. Now we define the following functions as

$$\mathcal{C}(w) = \int_0^w \cos\left(\frac{\pi w'^2}{2}\right) dw' \text{ and } \mathcal{S}(w) = \int_0^w \sin\left(\frac{\pi w'^2}{2}\right) dw'$$

so that

$$\int_0^w e^{i\pi w'^2/2} dw' = \mathcal{C}(w) + i\,\mathcal{S}(w)$$

Now we may expand all this to obtain

$$E_P = \frac{E_u}{2}[\mathcal{C}(u) + i\,\mathcal{S}(u)]_{u_1}^{u_2}[\mathcal{C}(v) + i\,\mathcal{S}(v)]_{v_1}^{v_2}.$$

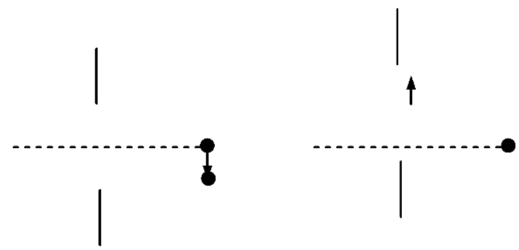
Carrying out the algebra

$$E_P = \frac{E_u}{2}\{[\mathcal{C}(u_2) + i\,\mathcal{S}(u_2) - \mathcal{C}(u_1) - i\,\mathcal{S}(u_1)] \times [\mathcal{C}(v_2) + i\,\mathcal{S}(v_2) - \mathcal{C}(v_1) - i\,\mathcal{S}(v_1)]\}$$

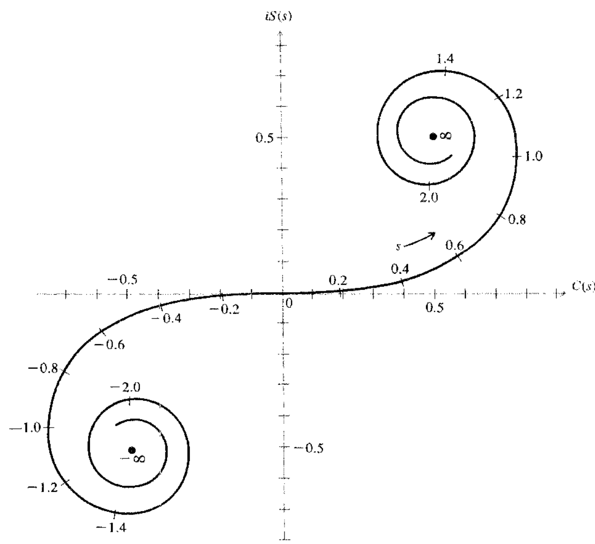
Therefore,

$$I_P = \frac{I_o}{4}\{[\mathcal{C}(u_2) - \mathcal{C}(u_1)]^2 + [\mathcal{S}(u_2) - \mathcal{S}(u_1)]^2\} \times \{[\mathcal{C}(v_2) - \mathcal{C}(v_1)]^2 + [\mathcal{S}(v_2) - \mathcal{S}(v_1)]^2\}.$$

It is not particularly useful to calculate only the irradiance on the axis of symmetry, so we need to find a way to calculate patterns off the symmetry axis. There are two ways of doing this. One method is to build into the diffraction integral an offset that allows us to do this directly, but it turns out that a better and more instructive method is to recognize that moving off axis is equivalent to relocating the aperture in the opposite direction to the way we moved our observation point. Here is a graph showing how this idea works. Then we will look at a graphical method of doing these calculations using a plot of $\mathcal{S}(w)$ versus $\mathcal{C}(w)$. This graph is called the Cornu spiral and is quite useful in visualizing what is going on with Fresnel diffraction from a rectangular aperture.



The left part of the figure shows that new relative location of the observation point with respect to the lower boundary of the aperture. The right side shows the observation point remainging fixed while moving the aperture up by the same distance the observation point was moved downward. The relative position of the observation point and the boundaries of the aperture are the same in each case. Using this approach allows us to visualize the process using the Cornu spiral. By the way, the Cornu spiral is just the vibration curve for the rectangular geometry as the spiral I showed for the circular aperture. Here is the Cornu spiral.




Before we start working with the Fresnel integrals and the Cornu spiral to analyze diffraction, let's look at a more general problem to understand better how these types of vibration curves work.

The general form of a one-dimensional diffraction integral is:

$$E(y) = \int A(y')\, e^{i\phi(y,\,y')}\, dy'$$

$A(y')$ is the aperture function (real), and $\phi(y, y')$ is the phase produced by the geometry and/or by a phase mask. Primed variables denote the aperture plane; unprimed variables denote the observation plane.

Define $$Z(u) = \int_0^u A(u')e^{i\phi(u')}du'$$

If $Z(u) = X(u) + iY(u)$, then

$$X(u) = \int_0^u A(u')\cos[\phi(u')]du' \qquad Y(u) = \int_0^u A(u')\sin[\phi(u')]du'$$

1. $ds = [dX^2 + dY^2]^{(1/2)} = A(u)\, du$ = differential arc length

2. Slope = $dY/dX = \tan[\phi(u)]$; independent of form of $\phi$(u)

3. Curvature = $K = d\phi/ds = [A(u)]^{-1}d\phi/du$

4. $$Z_{21} = Z_2 - Z_1 = \int_{u_1}^{u_2} A(u')e^{i\phi(u')}du' = \int_0^{u_2} A(u')e^{i\phi(u')}du' - \int_0^{u_1} A(u')e^{i\phi(u')}du'$$

5. A graph of $Y(u)$ versus $X(u)$ traces a vibration curve with $|Z_{21}|^2$ giving the square of the chord length between $(X_1, Y_1)$ and $(X_2, Y_2)$ which, in turn is proportional to the intensity of the diffraction pattern. $u$ is a parameter that is the rescaled arc length.

In your book, the aperture function is taken to be 1, so it simplifies things somewhat, but it also does not show the many interesting cases that can be used to relate diffraction to geometric figures. For what follows, I will use the aperture function to be one to make our life a bit easier. Let's have a look at how the Fraunhofer diffraction integral fits into this picture. For Fraunhofer diffraction, we have to identify the phase to be given by

$$\phi = ky\sin\theta \implies u = y,$$

And the curvature is given by $K = \frac{d\phi}{dy} = k\sin\theta$ with $R = \frac{1}{k\sin\theta}$. Therefore, when we are at the center of the pattern, the radius is infinite, meaning we have a straight line whose arc length is fixed by whatever the width of the aperture is. As $\sin\theta$ increases, the radius decreases with the arc length remaining constant until it closes into a circle at the first minimum. As $\sin\theta$ continues to increase, the radius get smaller until there is a maximum chord length between the ends of the curve.

The situation for Fresnel diffraction is different because the phase is a quadratic function instead of linear as in the Fraunhofer case. Now, the phase is given by

$$\phi(w') = \frac{\pi w'^2}{2} \text{ and } ds = dw'.$$

Therefore,

$$K = \frac{d\phi}{dw'} = \pi w' \ and \ R = \frac{1}{\pi w'}.$$

Now, as the arc length increases, *R* decreases. Because the Fresnel cosine and sine integrals are the X and Y components, the Fresnel integrals are the Cartesian coordinates of the Cornu spiral.

In the pages that follow, we work through several examples showing as many features of the calculations as we have time for. Got to the next page.

Summary of Cornu Spiral Plots

1. $\rho_o = 0.5$ m; $r_o = 1.0$ m, $\lambda = 500$ nm, $z_1 = -\infty$, $z_2 = \infty$, $y_1 = -0.75$ mm, $y_2 = 0.75$ mm
$u_1 = -2.6$, $u_2 = 2.6$, $v_1 = -\infty$, $v_2 = \infty$ $\quad$ $I/I_o = 0.91$

2. To make a plane wave incident, use $\rho_o = \infty$.
Then $u_1 = -1.5$ and $u_2 = 1.5$ $\quad$ $I/Io = 1.37$

3. Find $\lambda$ for maximum I. Note that maximum length occurs whenever $u_1 = -1.2$ and $u_2 = 1.2$. Because $u = y\ (2/\lambda r_o)1/2$, $\lambda = 781$ nm $\quad$ $I/Io = 1.78$

4. Off-axis example

5. Geometrical shadow

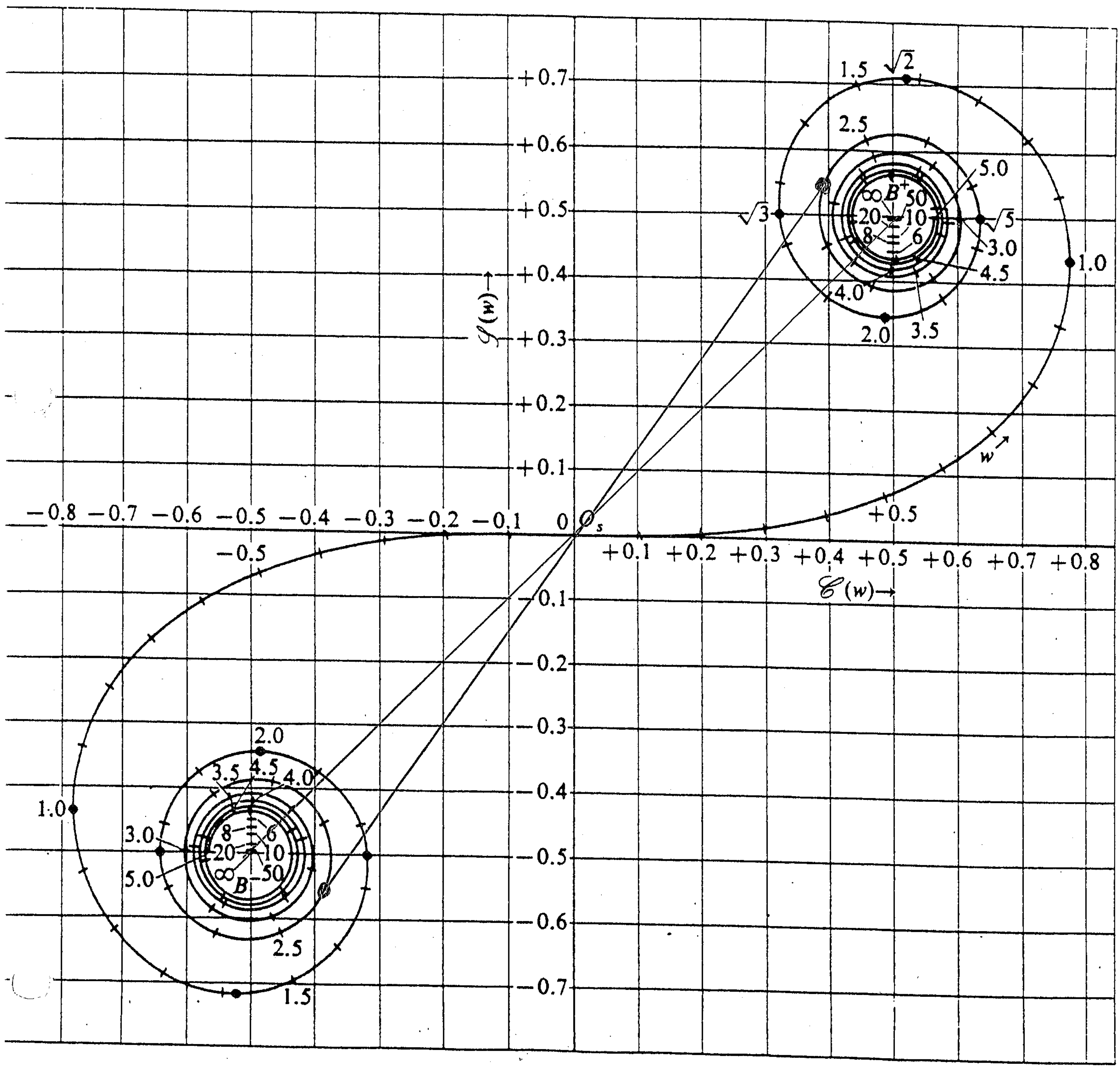
6. Minimum at center

7. Half-plane

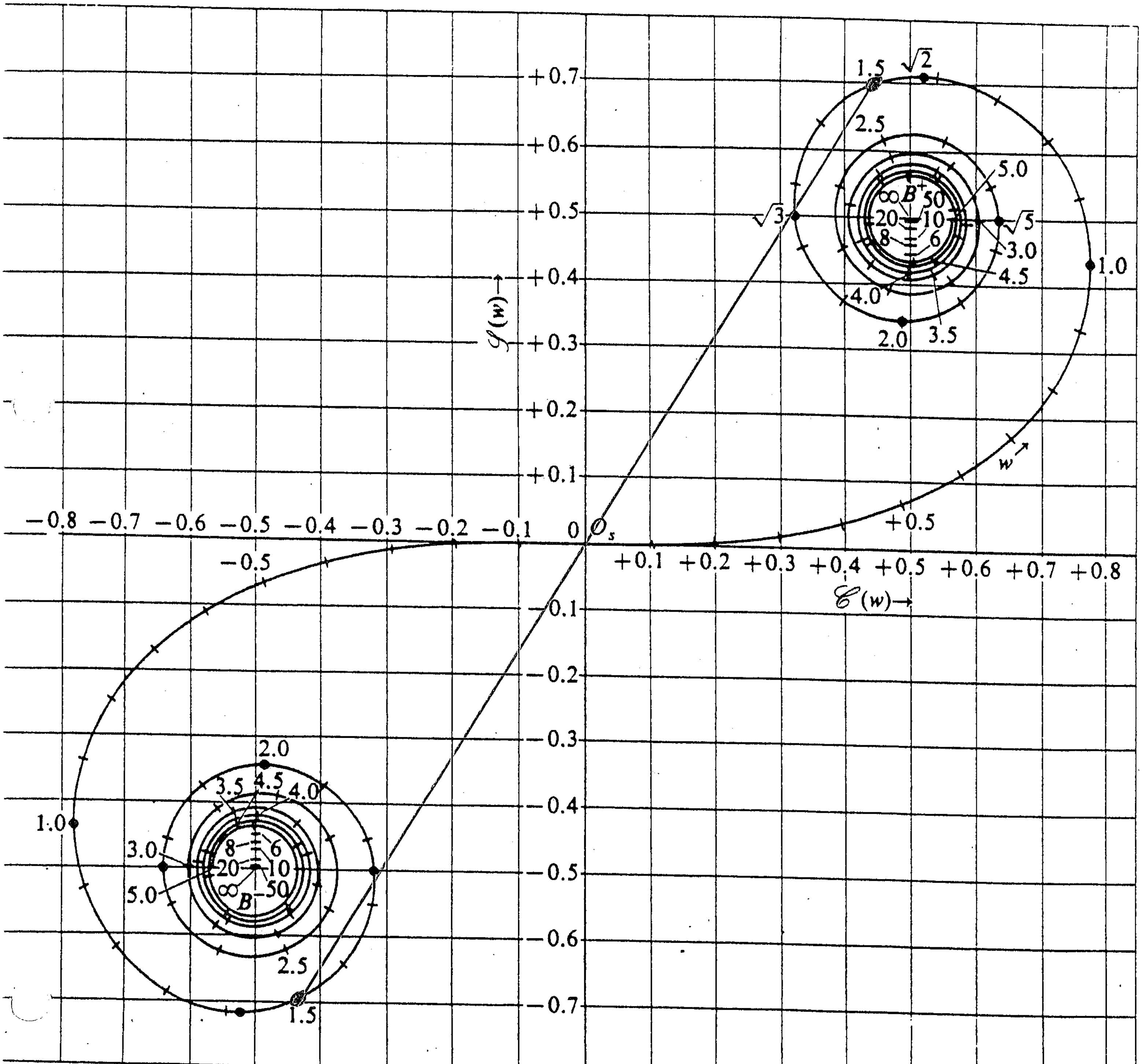
8. Barrier

#1

$$\frac{I}{I_0} = \left(\frac{16\,cm}{16.8\,cm}\right)^2 = 0.91 \text{ as we found}$$



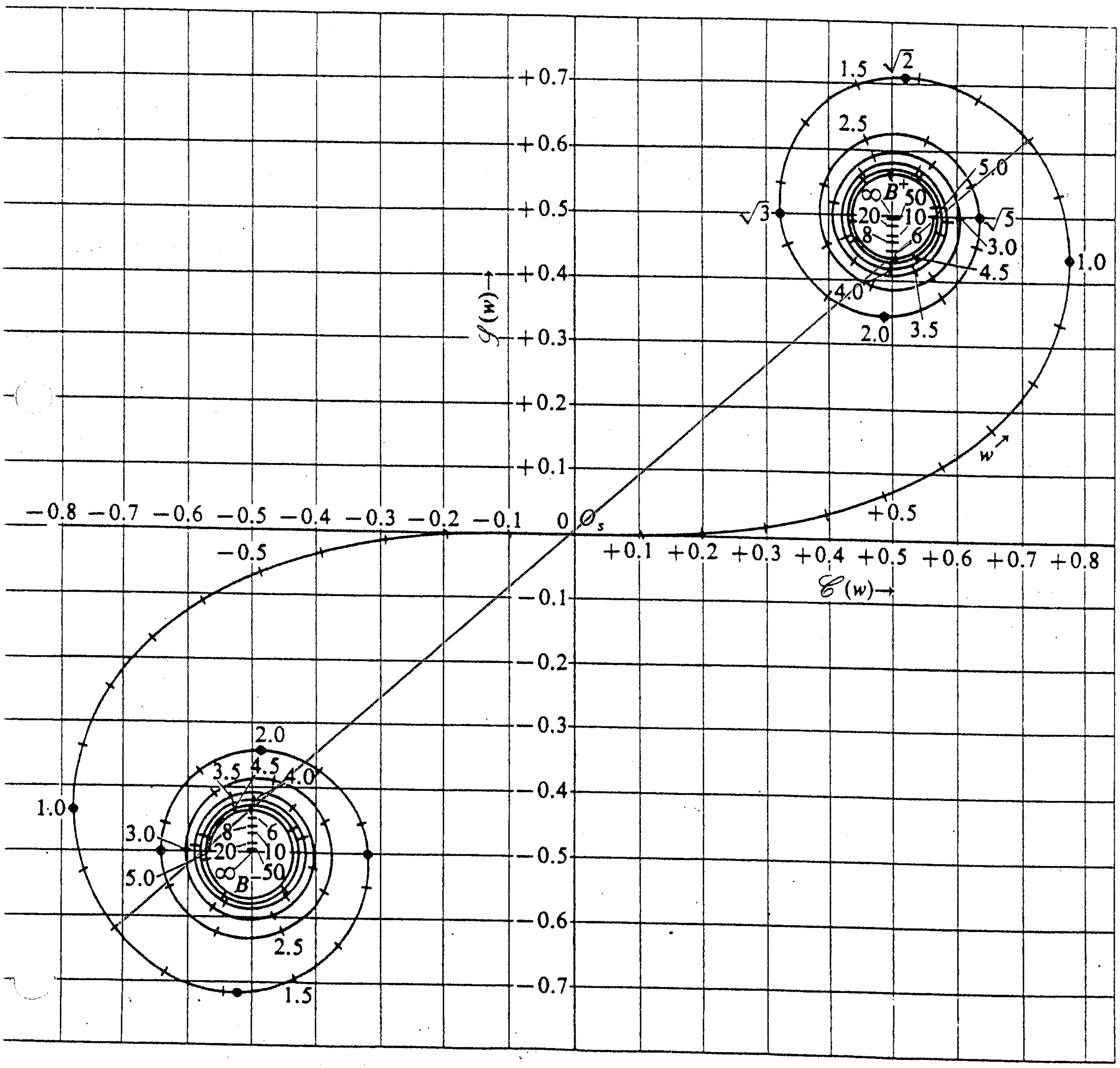
$$\frac{I}{I_0} = \left(\frac{19.7\,cm}{16.8\,cm}\right)^2 = 1.37 \text{ as before}$$



#3

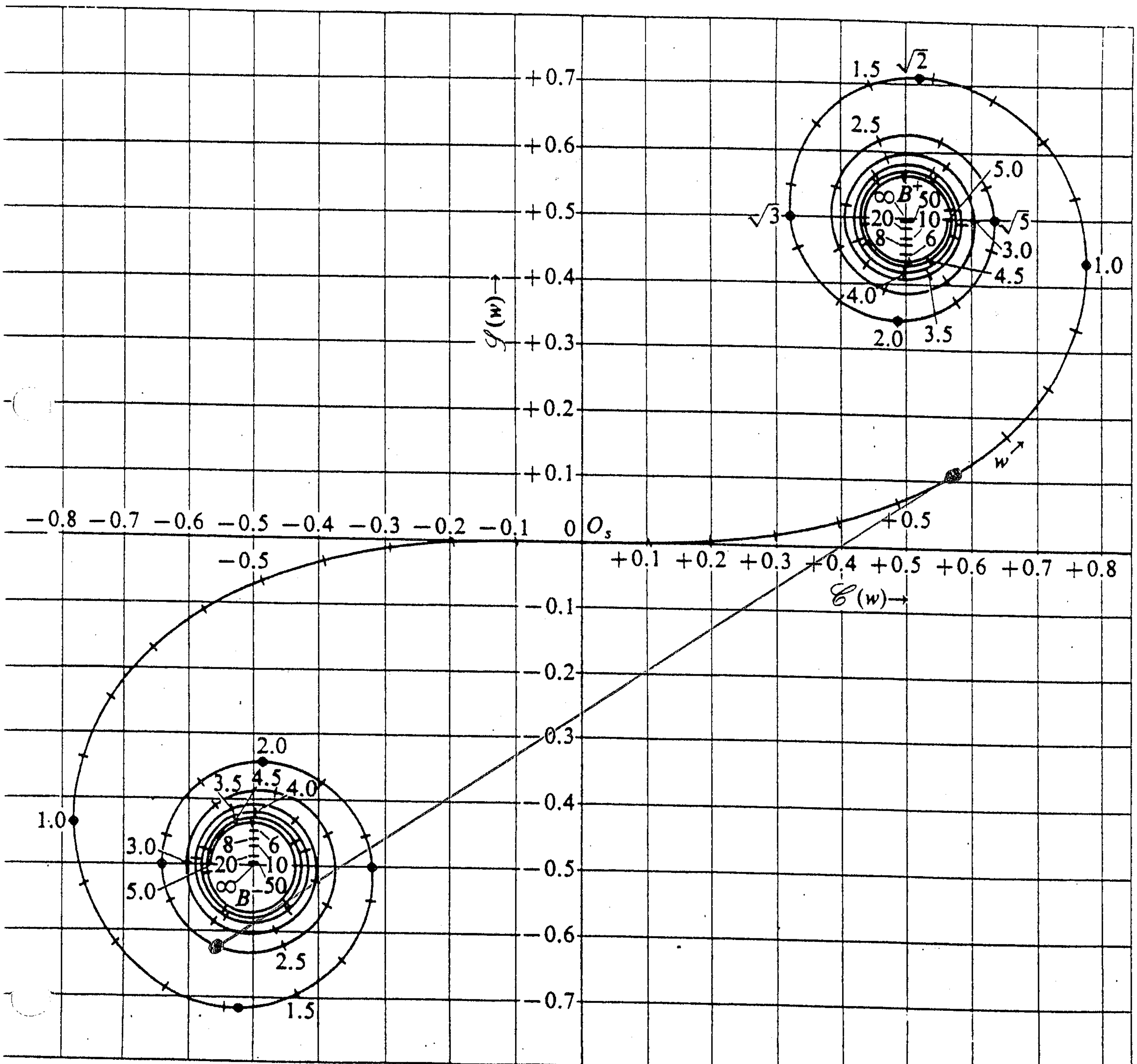
Max I : $\Delta u = 2.4;\ u_1 = -1.2;\ u_2 = 1.2$

$$\frac{I}{I_0} = \left(\frac{22.4\,cm}{16.8\,cm}\right)^2 = 1.78$$



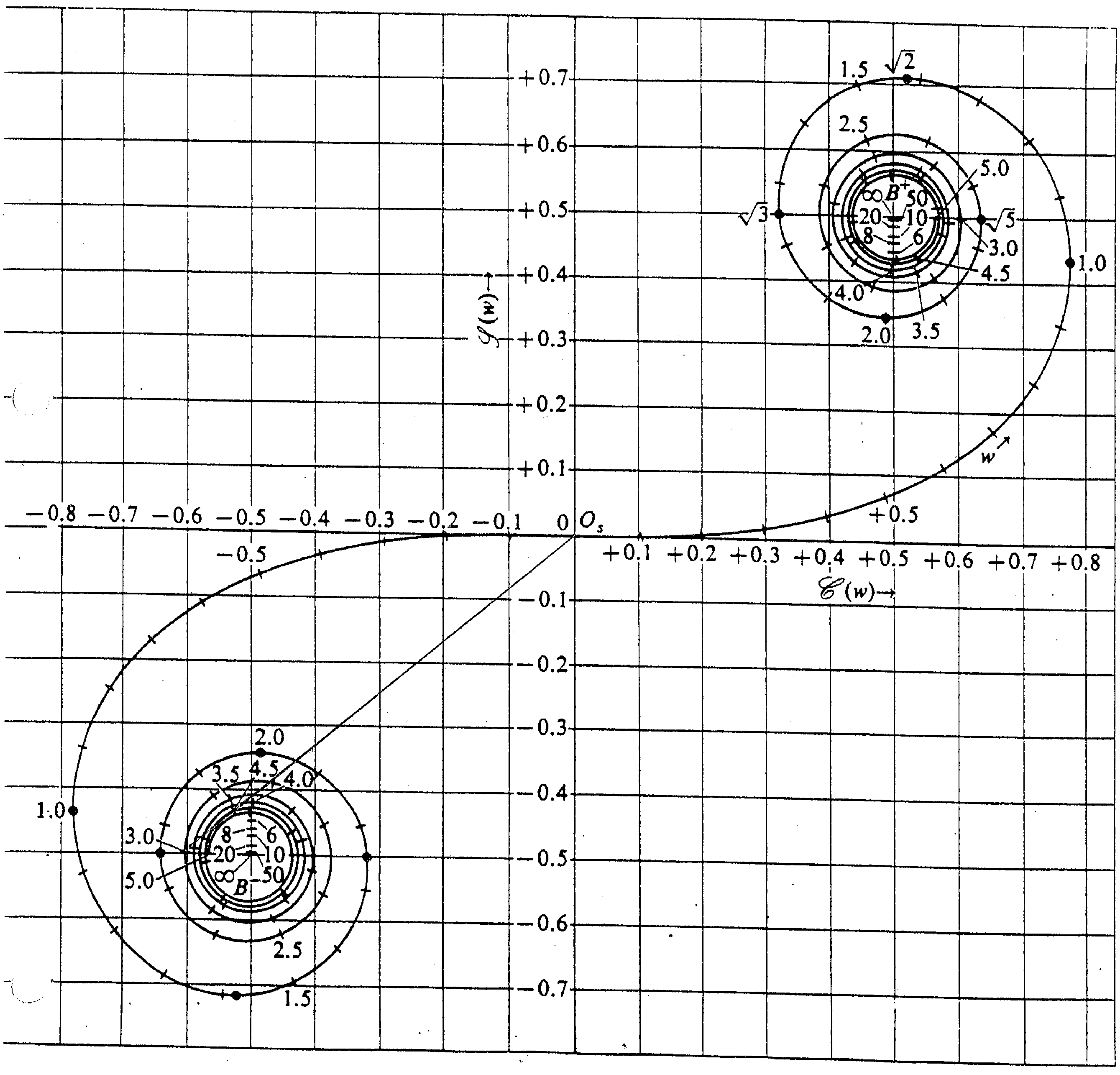
#4

$$\frac{I}{I_0} = \left(\frac{16\,cm}{16.8\,cm}\right)^2 = 0.91$$



#5

$$\frac{I}{I_0} = \left(\frac{9.2\,cm}{16.8\,cm}\right)^2 = 0.3$$



How a minimum occurs in the center of the pattern

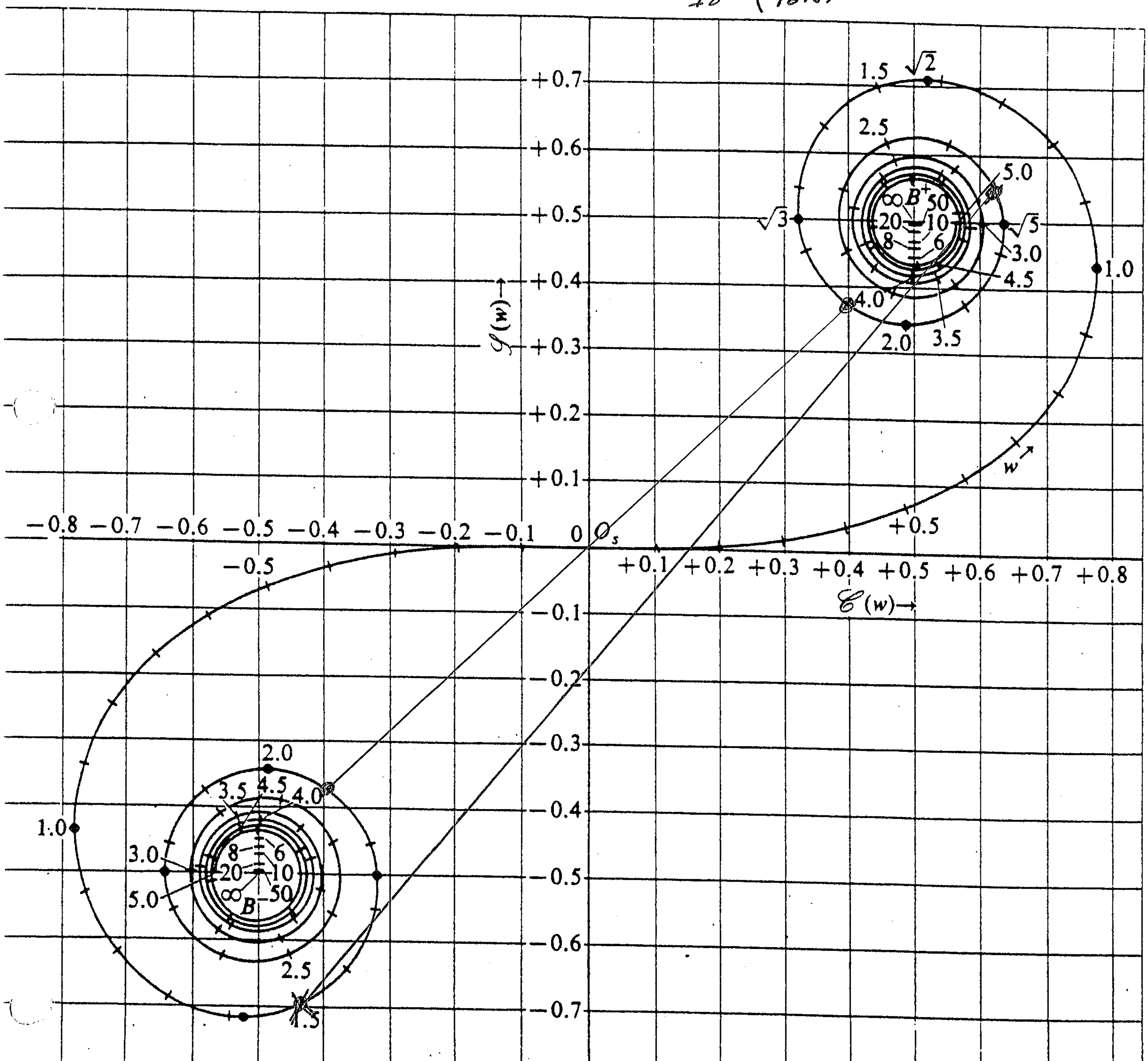
#6

$\Delta u = 3.8$ ; $u_1 = -1.9$ ; $u_2 = 1.9$

$\frac{I}{I_0} = \left(\frac{12.8 cm}{16.8 cm}\right)^2 = 0.58$

Moving off axis $\Rightarrow u_1 = -1.5$
$u_2 = 2.3$

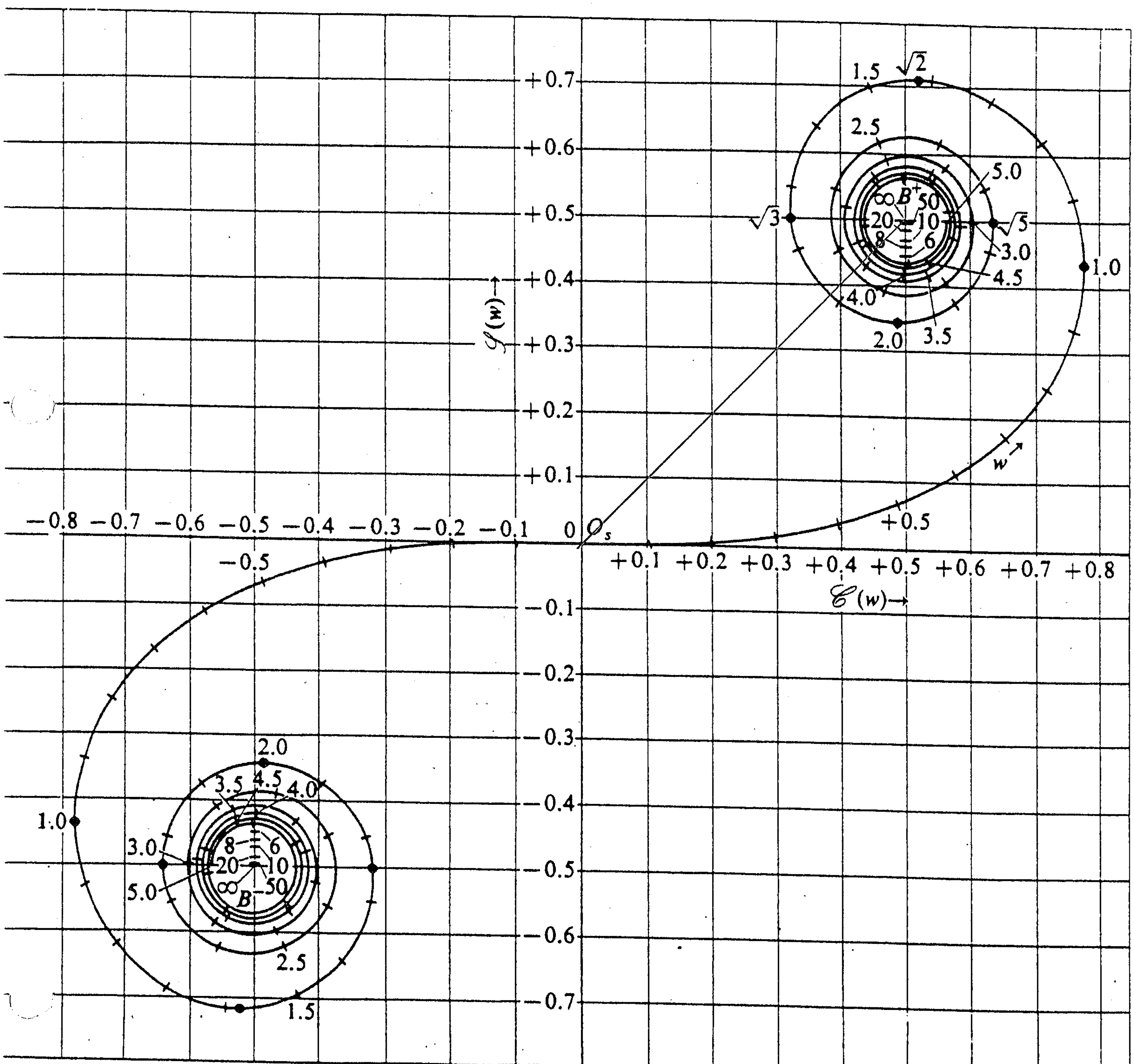
$\frac{I}{I_0} = \left(\frac{19.3}{16.8}\right)^2 = 1.32$



Half-Plane

#7

At edge $\frac{I}{I_0} = \left(\frac{8.4}{16.8}\right)^2 = 0.25$



Barrier width remains const.
Add contributions as vectors

#8