LAST TIME: Contour integration examples

What happens when we have poles on the real axis? Consider the integral given by

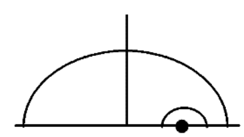
$$\int_{-\infty}^{+\infty} \frac{\sin kx}{x-2} dx$$

You notice that we have a simple pole at $x = 2$. There is even a question concerning the meaning of this integral because it is unbounded at the pole. We have to define what we mean, and we define the principal value of the integral by

$$P\int_{-\infty}^{+\infty} \frac{\sin kx}{x-2} dx = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{2-\epsilon} \frac{\sin kx}{x-2} dx + \int_{2+\epsilon}^{+\infty} \frac{\sin kx}{x-2} dx\right],$$

and we have avoided the singularity. Remember that we cannot have any poles on the contour when we use the residue theorem. We use the definition of the sine to split up our original integral into two parts given by

$$\int_{-\infty}^{+\infty} \frac{\sin kx}{x-2} dx = \frac{1}{2i}\left(\int_{-\infty}^{+\infty} \frac{e^{ikx}}{x-2} dx - \int_{-\infty}^{+\infty} \frac{e^{-ikx}}{x-2} dx\right).$$



We are assuming here that k is a positive number. To do the first integral, we close the contour in the top half, but avoid the pole by using a small semicircle to go around it. See the figure. From our previous theorems, Jordan's lemma, the large semicircle contributes nothing and because no poles are included, the integral is zero. For the second integral, we close the contour in the bottom half, but now the pole is inside the path. Just flip the large semicircle to the bottom without changing anything else. Now

$$K(2) = \lim_{z\to 2}(x-2)\frac{e^{-ikx}}{x-2} = e^{-2ik}.$$

Therefore, the integral is given by

$$\int_{C_{R\,below}} \frac{e^{-ikz}}{z-2} dz = -2\pi i e^{-2ik},$$

But we still must do the integral over the small semicircle and take the limit as ϵ, the radius of the small semicircle $\to 0$. We obtain

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} \frac{\sin kz}{z-2} dz = \lim_{\epsilon\to 0}\frac{1}{2i}\int_{\pi}^{0} \frac{\left[e^{ik(2+\epsilon e^{i\theta})} - e^{-ik(2+\epsilon e^{i\theta})}\right]}{\epsilon e^{i\theta}} \epsilon e^{i\theta} d\theta.$$

The integral goes from π to 0 because we are traveling the contour in the clockwise direction. We take the limit to obtain

$$\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{\sin kz}{z-2}dz = \frac{1}{2}\left(e^{i2k}-e^{-i2k}\right)\int_{\pi}^{0}d\theta = -i\pi\sin 2k.$$

We have defined the principal value to be the integral along the real axis up to the beginning of the small semicircle and continuing from the end of the small semicircle to infinity. Therefore,

$$P\int_{-\infty}^{+\infty}\frac{\sin kx}{x-2}dx - i\pi\sin 2k = \int_{-\infty}^{+\infty}\frac{\sin kx}{x-2}dx = \frac{1}{2i}\left(0 - -2\pi i e^{-2ik}\right).$$

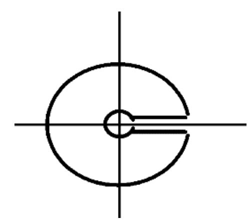
Finally

$$P\int_{-\infty}^{+\infty}\frac{\sin kx}{x-2}dx = \pi e^{-2ik} + \frac{\pi i}{2i}\left(e^{i2k}-e^{-i2k}\right) = \pi\cos 2k.$$

There is one last issue we need to consider. How do we deal with a branch point and a branch cut? Branch points and branch cuts occur when we have "multi-valued" functions. These include integrals of the form

$$\int_0^\infty x^p f(x)dx, \text{ where } p \text{ is a fraction.}$$

Remember that this is a problem because $z^p = r^p e^{ip\theta}$ is not analytic at $z = 0$ when p is not an integer. We must construct a closed contour that does not include the branch point and the branch cut. Here is a typical contour that would be used for a branch point at zero and a branch cut along the positive real axis. Suppose we look at the integral given by



$$I = \int_0^\infty \frac{\sqrt{x}}{x^2+1}dx.$$

There are two simple poles, one at i and one at $-i$. There are four separate integrals to do for this case. Imagine we use the contour shown and go in the ccw direction. Therefore, the total integral is given by

$$\oint_C \frac{\sqrt{z}}{z^2+1}dz = \int_{top\ of\ cut} + \int_{C_R} + \int_{bottom\ of\ cut} + \int_{C_\epsilon} \quad .$$

For our contour, the poles are inside the contour, so we need to get the residues. We have done this quite a few times, so I will state the results as follows.

$$Res\, f(+i) = \frac{\sqrt{i}}{2i} = \frac{e^{i\pi/4}}{2i} = \frac{\sqrt{2}}{4i}(1+i)$$

$$Res\, f(-i) = \frac{\sqrt{-i}}{-2i} = \frac{e^{i3\pi/4}}{-2i} = \frac{\sqrt{2}}{4i}(1-i)$$

$$\oint_C \frac{\sqrt{z}}{z^2+1}dz = 2\pi i\frac{2\sqrt{2}}{4i} = \pi\sqrt{2}.$$

The actual integral we want is along the top of the branch cut because it goes from 0 to infinity. We may use the same approach we have used before to see that the contribution of the integral along the big circle is zero. What about the small circle around the origin? Set $z = \epsilon e^{i\theta}$. Then

$$\int_{C_\epsilon} \frac{\sqrt{z}}{z^2+1}dz = \int_{2\pi}^{0} \frac{\sqrt{\epsilon}e^{i\theta/2}}{\epsilon^2 e^{2i\theta}+1}\epsilon i e^{i\theta}d\theta.$$

Simplify this and take the limit as ϵ goes to 0.

$$i\epsilon^{3/2}\int_{2\pi}^{0} \frac{e^{3i\theta/2}}{\epsilon^2 e^{2i\theta}+1}d\theta \longrightarrow 0 \; as \; \epsilon \longrightarrow 0.$$

Finally, what about the lower part of the branch cut? Along this part, $\theta = 2\pi$. Using $z = re^{i\theta}$, The integral becomes

$$\int_{\infty}^{0} \frac{\sqrt{re^{i2\pi}}}{r^2e^{i4\pi}+1}dr = -\int_{0}^{\infty} \frac{\sqrt{r}e^{i\pi}}{r^2+1}dr = \int_{0}^{\infty} \frac{\sqrt{r}}{r^2+1}dr = I.$$

Finally

$$\oint_C \frac{\sqrt{z}}{z^2+1}dz = \int_{top\ of\ cut} + \int_{C_R} + \int_{bottom\ of\ cut} + \int_{C_\epsilon} = 2I$$

and

$$I = \int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \frac{\pi\sqrt{2}}{2}.$$

There is an interesting relationship between the real and imaginary parts of a function that is analytic at z_o in the upper half plane. Use the Cauchy formula to express the value of f at z_o as follows

$$f(z_o) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_o}dz$$

with both C and z_o in the upper half plane. We assume that $|f(z)| \to 0$ as $z \to \infty$. Then we may let C be a large semicircle in the top half plane with its flat side along the real (x) axis. The integral along the curved part becomes zero, but we have to use our small semicircle of radius ε under the pole that will be on the x axis as z_o approaches the axis from the top. Therefore,

$$f(z_o) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(x)}{x-z_o}dx.$$

We use our definition of the principal value to obtain

$$f(x_o) = \frac{1}{2\pi i}\left[P\int_{-\infty}^{+\infty}\frac{f(x)}{x-x_o}dx + \lim_{\epsilon\to 0}\int_{-\pi}^{0}\frac{f(x_o+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}i\epsilon e^{i\theta}d\theta\right]$$

$$= \frac{1}{2\pi i}\left[P\int_{-\infty}^{+\infty}\frac{f(x)}{x-x_o}dx + i\pi f(x_o)\right].$$

We obtain for $f(x_o)$

$$f(x_o) = -\frac{i}{\pi}P\int_{-\infty}^{+\infty}\frac{f(x)}{x-x_o}dx.$$

However, $f(x)$ has both real and imaginary parts, so we may write

$$Re[f(x_o)] = \frac{1}{\pi}\ P\int_{-\infty}^{+\infty}\frac{Im[f(x)]}{x-x_o}dx$$

and

$$Im[f(x_o)] = -\frac{1}{\pi}\ P\int_{-\infty}^{+\infty}\frac{Re[f(x)]}{x-x_o}dx$$

This means that the real and imaginary part of a function that satisfies these conditions are linked. In materials, one is accustomed to considering the real and imaginary part of the dielectric tensor or the conductivity tensor. The connections between the real and imaginary parts of either are called the Kramers-Kronig relations. They mean that we may measure the absorption and get the dispersion or vice-versa. Thes are very important in optical measurements and spectroscopy. Essentially, anytime we have waves involved, there will be dispersion and absorption, so these relations show up in many areas of physics.

Return to the previous Green function problem – damped, driven harmonic oscillator.

Recall that we looked at the Green function for a harmonic oscillator, but we were temporarily stumped because we did not know how to transform back from the omega representation to the time representation. Now we know how to do that, so let's see how it works.

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F(t).$$

To put the equation in a more standard form, let's divide through by m and define some new constants which should be familiar to you.

$$\frac{d^2x(t)}{dt^2} + 2\gamma\frac{dx(t)}{dt} + \omega_o^2x(t) = f(t).$$

We will have some initial conditions on this equation, and we will worry about those later. The Green function we need to find satisfies the equation given by

$$\frac{d^2G(t,t')}{dt^2} + 2\gamma\frac{dG(t,t')}{dt} + \omega_o^2 G(t,t') = \delta(t-t').$$

Equations of this type may be solved using a Fourier transform to convert the ODE to an algebraic equation, much in the same way you used the substitution of e^{qt} to convert the homogeneous version to an algebraic equation. The Fourier transform, however, is useful to convert from one space to another, such as time to frequency, space to k-space in solid state physics, or space to spatial frequency in optics. Here are the two most common forms of the transform and its inverse transform.

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt \; and \; f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\,d\omega,$$

where $f(t)\ and\ F(\omega)$ are Fourier transform pairs. Sometimes, it is convenient to write

$$F(\omega) = \mathcal{F}[f(t)] \text{ and } f(t) = \mathcal{F}^{-1}[F(\omega)].$$

The use of the factor $\sqrt{2\pi}$ is not standard. Some authors use 2π in only one of the factors, but the product must be 2π for one-dimensional transforms. In other applications, we might right

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}\,dx \text{ and } f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(k)e^{ikx}\,dk.$$

The Fourier transform of this equation is given by

$$-\omega^2 G(\omega,t') - 2i\,\gamma\omega G(\omega,t') + \omega_o^2 G(\omega,t') = \frac{1}{\sqrt{2\pi}}e^{i\omega t'}.$$

We solve for $G(\omega,t')$ to obtain

$$G(\omega,t') = \frac{1}{\sqrt{2\pi}}\frac{e^{i\omega t'}}{\omega_o^2 - 2i\,\gamma\omega - \omega^2}.$$

Now we take the inverse transform to get

$$G(t,t') = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{e^{i\omega t'}}{\omega_o^2 - 2i\,\gamma\omega - \omega^2}e^{-i\omega t}d\omega.$$

As you can see, this is a relative straightforward way to obtain the Green function in integral form, but we must use the residue theorem to do the integral. The integrand has two simple poles located at

$$-i\gamma \pm \sqrt{\omega_o^2 - \gamma^2}.$$

We have not yet done contour integration, but the result is given by

$$G(t,t') = 0 \quad t < t' \text{ and } G(t,t') = e^{-\gamma(t-t')}\frac{\sin[\Omega(t-t')]}{\Omega} \quad t > t'.$$

So how do we get this result using our recently developed ideas of contour integration?

Both of these poles are in the lower half plane, symmetrically located with respect to the y axis. Rewrite our integral as

$$G(t,t') = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega(t'-t)}}{\omega_o^2 - 2i\,\gamma\omega - \omega^2} d\omega.$$

This integral has exactly the same form as we used to prove Jordan's lemma earlier. Here, ω is the integration variable instead on z, and $(t'-t)$ plays the role of k. For $(t'-t) > 0$ or $t < t'$, we close the contour upward and find no poles, so $G(t,t') = 0$ for $t < t'$as required by causality. For $t > t'$, we close to the bottom – see Theorem II number 2 from lecture 22. Here, then are the poles, so we set $\Omega = \sqrt{\omega_o^2 - \gamma^2}$ and get the following given by

$$G(t,t') = -\frac{1}{2\pi}(-2\pi i)\left(\frac{e^{i(t'-t)(-i\gamma+\Omega)}}{2\Omega} + \frac{e^{i(t'-t)(-i\gamma-\Omega)}}{2\Omega}\right) = e^{-\gamma(t-t')}\frac{\sin\Omega(t-t')}{\Omega}, t > t'$$

Keep in mind that we now have solved the problem for any $f(t)$. We need only solve the integral given by

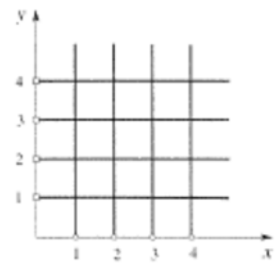
$$x(t) = \int_0^t f(t')G(t,t')dt'$$

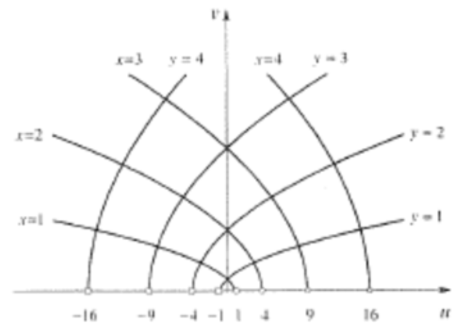
Conformal mapping

Recall that we found some rather elementary mappings that accomplished translation, rotation, stretching, inversion, and combinations of those. Here they are again.

Translation: $w = z + \beta$
Rotation: $w = ze^{i\theta_o}$
Stretching: $w = az$
Inversion: $w = \frac{1}{z}$
Linear Transformation: $w = az + \beta$ (translation, rotation, stretching)
Bilinear or fractional Transformation: $w = \frac{az+\beta}{\gamma z+\delta}$ (translation, rotation, stretching, inversion)

We specifically looked at how the transformation (or mapping) given by $w = f(z) = z^2$ appears in the $u - v$ plane when we consider a particular set of curves in the $x - y$ plane. Recall that we are using $w = f(z) = u(x,y) + iv(x,y)$. We looked at a set of curves $x = 1, x = 2, x = 3, x = 4, y = 1, y = 2, y = 3,$ and $y = 4$. Just to see how one looks, consider $x = 1$, where y extends from 0 to ∞. For this function, $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. Therefore, $u(x,y) = 1 - y^2$ and $v(x,y) = 2y$. So, $u = 1 - \frac{v^2}{4}$. Similar shapes occur for the other straight lines. Here are the graphs of each set of curves in the $x - y$ plane and in the $u - v$ plane.





Notice that these functions $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ satisfy the Cauchy-Riemann relations because

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

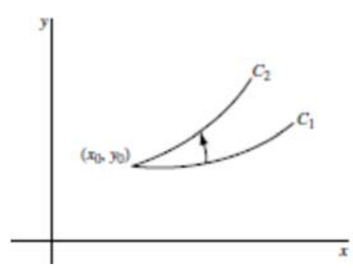
and therefore,

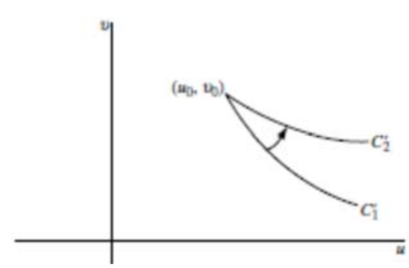
$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 2x \text{ and } \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y} = - - 2y.$$

We also saw that we could derive the two-dimensional Laplace's equation for $u(x,y)$ and $v(x,y)$ to obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

All of these findings suggest that complex variables and mapping may have some useful applications is solving two-dimensional potential problems as well as two-dimensional problems in other fields as well, *e.g.*, heat flow and fluid flow. There are four very important theorems concerning these issues. First, if $f(z)$ is analytic in a region R and $f'(z) \neq 0 \; in \; R$, then the mapping $w = f(z)$ is conformal at all points in R. Recall that conformal means that the mapping preserves both angles and the sense of the angles as shown in the figure. Second, if $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R, then the family of curves given by





$$u(x,y) = C_1 \text{ and } v(x,y) = C_2$$

are orthogonal. To prove this, suppose two particular members of these families given by

$$u(x,y) = u_o \text{ and } v(x,y) = v_o$$

intersect at the point (x_o, y_o). Note that $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$ so that $\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$. The calculation for $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$ so that $\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$. These two expressions represent the slopes of the respective curves as the point of intersection (x_o, y_o). We may use the Cauchy-Riemann relations to write these as $\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}}$ and $\frac{dy}{dx} = -\frac{-\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial y}}$. Multiplying the two slopes gives

$$\left(-\frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}}\right)\left(-\frac{-\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial y}}\right) = -1,$$

which is the condition for orthogonality. Third, the harmonic nature of the equations for $u(x,y)$ and $v(x,y)$ as functions of x and y carries over to functions $U(u,v)$ and $V(u,v)$ in the $w-$plane. This preservation of Laplace's equation is proved by using straightforward chain rule differentiation and the Cauchy-Riemann relations. If the mapping function is analytic and f' is not zero, the mapping is one to one. Fourth, Riemann's mapping theorem states that if C is a simple closed curve in the z – plane forming the boundary of a simply connected region R, and if C′ is a circle of radius one and center at the origin forming the boundary of region R′ in the w - plane, every point in R′ is the image of a point in R under a function $w = f(z)$. Here is a figure that represents this statement. Although the mapping theorem guarantees the existence of such a mapping, it does not tell us how to find it. The mapping theorem may also be extended to a region bounded by two simple closed curves, one inside the other, where these curves map onto a region bounded by two concentric circles.

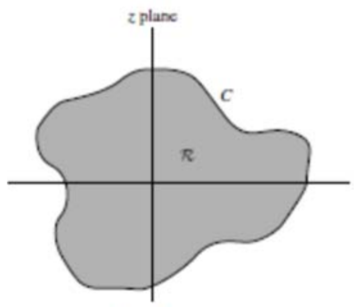


Fig. 8-3

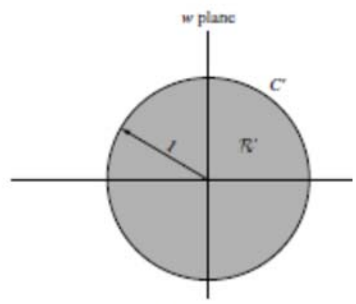


Fig. 8-4

NEXT TIME: Conformal mapping continued