Green function for the wave equation with sources.

$$\nabla^2\psi(\mathbf{r},\mathrm{t}) - \frac{1}{c^2}\frac{\partial^2\psi(\mathbf{r},\mathrm{t})}{\partial t^2} = f(\mathbf{r},\mathrm{t})$$

We use the usual method by getting the Green function for an impulse source and then using superposition to get the wave function. Therefore,

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(\mathbf{r},t;\mathbf{r}',t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t').$$

so that

$$\psi(r,t) = \int d^3\,r' \int dt'\; G(\mathbf{r},t;\mathbf{r}',t') f(\mathbf{r}',\mathrm{t})$$

Fourier transform each side and execute the operators ∇^2 and $\frac{\partial^2}{\partial t^2}$ to obtain

$$G(\mathbf{r},t;\mathbf{r}',t') = \int d^3\,k \int d\omega\; e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\, e^{-i\omega(t-t')}\, g(\mathbf{k},\omega),$$

$$\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') = \frac{1}{(2\pi)^4}\int d^3\,k \int d\omega\; e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\, e^{-i\omega(t-t')}$$

and

$$g(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\frac{1}{k^2 - \frac{\omega^2}{c^2}}.$$

Now we must transform $g(\mathbf{k},\omega)$ back to ($\mathbf{r}$,t) space. Therefore,

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{(2\pi)^4}\frac{\int d^3\,k \int d\omega\; e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\, e^{-i\omega(t-t')}}{k^2 - \frac{\omega^2}{c^2}}.$$

This integral has poles on the real axis located at positions $k = \pm\frac{\omega}{c}$. Using the Jordan lemma, we close the contour above the axis for $t < t'$ and choose our contour to exclude the poles here so that $G = 0$ to satisfy causality. For the lower contour, both poles are inside the contour so the integral becomes

$$\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t-t')}}{(k-\omega/c)(k+\omega/c)} = \frac{2\pi c}{k}\sin ck(t-t').$$

We do the integral over k in spherical coordinates with $d^3k = k^2 \sin\theta\; d\theta\; d\phi$ and $\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') = k|\mathbf{r}-\mathbf{r}'|\cos\theta$ to obtain

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{(2\pi)^4}\int d^3k\; e^{ik|\mathbf{r}-\mathbf{r}'|\cos\theta}\,\frac{2\pi c}{k}\sin ck(t-t').$$

Simplifying by doing the integral over ϕ and rearranging gives

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_0^{\pi} k^2\,dk\,(\sin\theta\,d\theta)\,e^{ik|\mathbf{r}-\mathbf{r}'|\cos\theta}\,\frac{c}{k}\sin ck(t-t')$$

$$= \int_0^{\infty}\frac{2}{(2\pi)^2|\mathbf{r}-\mathbf{r}'|}\sin ck(t-t')\ \sin k|\mathbf{r}-\mathbf{r}'|\,dk.$$

Now use two delta-function identities given by

$$\delta(x-a) = \frac{2}{\pi}\int_0^{\infty}\sin kx\sin ka\,dk \ \text{ and } \ \delta(ax) = \frac{\delta(x)}{|a|}$$

to obtain

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\delta\left[t' + \frac{|\mathbf{r}-\mathbf{r}'|}{c} - t\right].$$

Finally,

$$\psi(\mathbf{r},t) = \int d^3r'\int dt'\,\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\delta\left[t' + \frac{|\mathbf{r}-\mathbf{r}'|}{c} - t\right] f(\mathbf{r}',t')$$

and

$$\psi(\mathbf{r},t) = \frac{1}{4\pi}\int d^3r'\,\frac{f\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}.$$

The value $t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}$ is the retarded time. When you solve this problem in other contexts, the function f will take on values of different physical parameters. ψ will be replaced with other variables as well.