LAST TIME:

$$res\ f\ at\ a = \frac{1}{2\pi i}\oint_C f(z)dz,$$

$$\oint_C f(z)dz = 2\pi i \sum_{n=1}^{N} Res\ f(z_n) = 2\pi i \sum_{n=1}^{N} K_n,$$

There are numerous methods of calculating these residues, and I list them below.

1. We may calculate the Laurent series and pick out the coefficient c_{-1} .

2. For a simple pole, the residue at a is given by

$$Res\ f(a) = K_n(a) = \lim_{z\to a}(z-a)\,f(z).$$

Number 2 is true because if you substitute the Laurent series in for $f(z)$, it yields only c_{-1} .

3. For a pole of order *m*, we may calculate the residue by considering the Laurent series once again and differentiate it term by term, and we obtain

$$Res\ f(a) = K_n(a) = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)].$$

One use of the residue theorem is to calculate the integral of functions having the form

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta.$$

To see how this works with complex integration, we need to convert the sine function, cosine functions, and $d\theta$ to their complex forms. We use $z = re^{i\theta}, dz = ire^{i\theta}d\theta = ie^{i\theta}d\theta$ if we evaluate the integrals around the unit circle in the complex plane. Therefore, $d\theta = \frac{dz}{iz}$.

We express sine and cosine in terms of *z* as follows.

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

and

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} = \frac{1}{2i}\left(z-\frac{1}{z}\right).$$

Using these transformations, we may convert trigonometric functions to functions of *z* and use contour integration to obtain useful results.

Let's look at some examples of how to use the residue theorem for integrals that don't appear to be quite so straightforward.

Example 1

Evaluate

$$\int_0^{\pi} \frac{1}{3+\sin^2\theta}\,d\theta.$$

The first thing we notice is that the integral does not run over the full range we need to close the contour. We note, however, that $\sin^2\theta$ is an even function, so we may write

$$\int_{-\pi}^{\pi} \frac{1}{3+\sin^2\theta}\,d\theta = \int_{-\pi}^{0} \frac{1}{3+\sin^2\theta}\,d\theta + \int_0^{\pi} \frac{1}{3+\sin^2\theta}\,d\theta$$

$$= 2\int_0^{\pi} \frac{1}{3+\sin^2\theta}\,d\theta.$$

Now use the results above to convert this integral to one over z.

$$\int_0^{\pi} \frac{1}{3+\sin^2\theta}\,d\theta = \frac{1}{2}\oint_{unit\ circle} \frac{1}{3+\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^2}\frac{dz}{iz}$$

$$= \frac{1}{2i}\oint_{unit\ circle} \frac{1}{\left[\frac{7}{2}-\frac{1}{4}z^2-\frac{1}{4z^2}\right]}\frac{dz}{z}.$$

Eliminate the fractions in the denominator and rationalize to obtain

$$\int_0^{\pi} \frac{1}{3+\sin^2\theta}\,d\theta = 2i\oint_{unit\ circle} \frac{z}{z^4-14z^2+1}\,dz.$$

Now we find the residues to see how many lie in the unit circle by solving the quadratic equation in z^2.

$$z_p^4 - 14z_p^2 + 1 = 0$$

We solve the quadratic equation to obtain

$$z_p^2 = \frac{14 \pm \sqrt{14^2-4}}{2} = 7 \pm \frac{1}{2}\sqrt{192} = 7 \pm 4\sqrt{3} = 7 \pm 6.93.$$

The four roots are

$$z_{1,2} = \pm 3.73\ and\ z_{3,4} = \pm 0.27$$

Only $z_{3,4}$ are inside the unit circle, so we do not need to worry about $z_{1,2}$.
Therefore,

$$K(z_3) = \lim_{z \to z_3}(z - z_3)\frac{z}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} = \frac{z_3}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)}$$

and

$$K(z_4) = \lim_{z \to z_4}(z - z_4)\frac{z}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} = \frac{z_4}{(z_4 - z_1)(z_4 - z_2)(z_4 - z_3)}.$$

We may use the fact that $z_2 = -z_1$ and $z_4 = -z_3$ to obtain the sum of the residues inside the unit circle as

$$K(z_3) + K(z_4) = \frac{z_3}{(z_3^2 - z_1^2)(z_3 - z_4)} + \frac{z_4}{(z_4^2 - z_1^2)(z_4 - z_3)} = \frac{1}{(z_4^2 - z_1^2)}$$

$$= \frac{1}{\left(7 - 4\sqrt{3}\right) - \left(7 + 4\sqrt{3}\right)} = -\frac{\sqrt{3}}{24}.$$

We use the residue theorem to obtain

$$\int_0^{\pi} \frac{1}{3 + \sin^2\theta}\, d\theta = 2\pi i(2i)\left(-\frac{\sqrt{3}}{24}\right) = \frac{\sqrt{3}}{6}\pi. \text{ (Whew!!).}$$

There is a really nice example of calculating the flux through a circular loop outside a long, straight wire carrying a time-dependent current $\frac{dI}{dt}$ that I will assign as homework. It is actually a good physics problem that uses the same ideas as this example does.

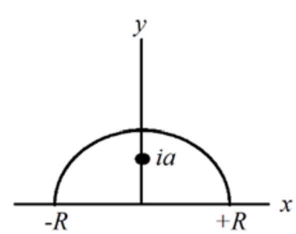
Other integrals of the form $\int_{-\infty}^{+\infty} f(x)dx$ may also be converted into an integral in the complex plane so that we may use these techniques. For these problems, we will close the contour by adding addition sections that do not contribute to the integral's value.

Example 2

Consider the integral given by

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2 + a^2)^2}\, dx.$$

One way to think of this integral is along the real axis in the complex plane. To make a closed contour that is a requirement for using the residue theorem, we will have to close our contour by taking a large semi-circle whose radius becomes infinite. We have to make sure that our integral does not have a value along that circle, or the technique does not work. In other words, the value of our integral as $R \to \infty$ must go to zero. We are essentially doing the following



$$\oint_C \frac{1}{(z^2+a^2)^2} = \lim_{R\to\infty}\left[\int_{-R}^{+R}\frac{1}{(z^2+a^2)^2}dx + \int_{semi-circle}\frac{1}{(z^2+a^2)^2}.\right]$$

The issue to settle is how our function behaves on the large semicircle. If we show that the contribution to the integral is zero, then this approach is fine. Consider the absolute value of the integrand.

$$\left|\frac{1}{(z^2+a^2)^2}\right| = \frac{1}{|z|^4}\frac{1}{\left|1+\frac{a^2}{z^2}\right|^2} \le \frac{1}{|z|^4}\frac{1}{\left(1-\frac{|a|^2}{|z|^2}\right)^2}$$

The last step makes use of the relationship we gave earlier given by

$$|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$$

and $z_1 + z_2 = z_1 - (-z_2)$. We may make the semicircle large enough so that $\frac{|a|}{|z|} < \frac{1}{\sqrt{2}}$, and this gives

$$\left|\frac{1}{(z^2+a^2)^2}\right| \le \frac{4}{|z|^4}.$$

This value of $\frac{4}{|z|^4}$ sets the maximum value of the integrand. We multiply this number by the length of the contour to see what its limit is as $R\to\infty$. If that number is zero, then this process works. Therefore,

$$\pi R\left(\frac{4}{R^4}\right) = \frac{4\pi}{R^3},$$

which goes to zero as $R\to\infty$, so everything is fine. Finally, the value of our integral is given by

$$\oint_C \frac{1}{(z^2+a^2)^2} = \int_{real\ axis}\frac{1}{(z^2+a^2)^2}dz.$$

There are two order 2 poles given by $z = \pm ia$. Because we closed the contour in the top half, only the $z = +ia$ pole contributes. Now, we calculate the residue of the order 2 pole using our expression given by

$$Res\, f(a) = K_n(a) = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)],$$

so

$$K(ia) = \lim_{z\to ia}\frac{1}{1!}\frac{d}{dz}\left[(z-ia)^2\frac{1}{(z^2+a^2)^2}\right] = \lim_{z\to ia}\frac{d}{dz}\frac{1}{(z+ia)^2}$$

$$= \lim_{z\to ia}\frac{-2}{(z+ia)^3} = \frac{-2}{(2ia)^3} = \frac{1}{4ia^3}.$$

Finally, the integral is

$$\oint_C \frac{1}{(z^2+a^2)^2} = 2\pi i \frac{1}{4ia^3} = \frac{\pi}{2a^3} = \int_{-\infty}^{+\infty} \frac{1}{(x^2+a^2)^2} dx.$$

What I have used in this example is a result that has a more general result. We know that

$$\int_C f(z)\, dz \le \int_C |f(z)|\,|dz| \le ML,$$

where M is the upper bound of $|f(z)|$ on C and L is the length of C. We want to prove that if

$|f(z)| \le \frac{M}{R^k}$ for $z = Re^{i\theta}$ with k > 1, $\lim_{R\to\infty} \int_C f(z)\, dz = 0$. Here C is the semicircle used before.

$$\left|\int_C f(z)\, dz\right| \le \int_C |f(z)|\,|dz| \le \frac{M}{R^k}\pi R = \frac{\pi M}{R^{k-1}}, \text{which goes to zero in the limit of } R \to \infty.$$

This is a useful result because it tells us that as long as the denominator has at least a positive power, our method works.

I suspect that you are all of you are familiar with Fourier transforms that typically have the form

$$F(k) \sim \int_{-\infty}^{+\infty} f(x) e^{ikx}\, dx.$$

There is another theorem that is useful for evaluating such integrals. If $F(z) \le \frac{M}{R^k}$ for $z = Re^{i\theta}$,

$\lim_{R\to\infty} \int_C e^{imz} F(z)\, dz = 0$ for $k > 0$ and M is a constant. Here, m is a positive constant and C is the semicircular arc used before. Written in terms of θ, our integral becomes

$$\int_0^{\pi} e^{imRe^{i\theta}} F\left(Re^{i\theta}\right) iRe^{i\theta} d\theta$$

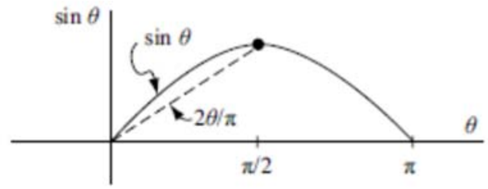
and

$$\left|\int_0^{\pi} e^{imRe^{i\theta}} F\left(Re^{i\theta}\right) iRe^{i\theta} d\theta\right| \le \int_0^{\pi} \left|e^{imRe^{i\theta}} F\left(Re^{i\theta}\right) iRe^{i\theta}\right| d\theta$$

$$= \int_0^{\pi} \left|e^{imR\cos\theta - mR\sin\theta} F\left(Re^{i\theta}\right) iRe^{i\theta}\right| d\theta = \int_0^{\pi} e^{-mR\sin\theta} \left|F\left(Re^{i\theta}\right)\right| R d\theta$$

$$\le \frac{M}{R^{k-1}} \int_0^{\pi} e^{-mR\sin\theta}\, d\theta = \frac{2M}{R^{k-1}} \int_0^{\frac{\pi}{2}} e^{-mR\sin\theta}\, d\theta$$

Now use the fact that $\sin\theta \geq \frac{2\pi}{\theta}$ for $0 \leq \theta \leq \frac{\pi}{2}$ as is seen from the graph. This means that the integral above is related to another integral by



$$\frac{2M}{R^{k-1}}\int_0^{\frac{\pi}{2}} e^{-mR\sin\theta}\, d\theta \;\; \leq \frac{2M}{R^{k-1}}\int_0^{\frac{\pi}{2}} e^{-2mR\theta/\pi}\, d\theta$$

$$= \frac{\pi M}{R^k}(1 - e^{-mR}),$$

which goes to zero when R goes to infinity. This is known as Jordan's lemma.

Here is an example of how it works. Consider

$$\int_{-\infty}^{+\infty} e^{imx}\frac{dx}{a^2 + x^2}$$

where $m \geq 0 \; and \; a > 0$. We consider closing the contour in the upper half plane as usual. Therefore, consider

$$\int_C e^{imz}\frac{dz}{a^2 + z^2}$$

with a pole in the upper plane at $z = ia$. The residue is given by

$$\left(\frac{e^{imz}}{z + ia}\right)_{z=ia} = \frac{1}{2ia}e^{-ma}$$

and

$$\int_{-\infty}^{+\infty} e^{imx}\frac{dx}{a^2 + x^2} = \frac{\pi}{a}e^{-ma}.$$

Here is a summary of some of the important proofs involving closing the contour in these cases. The reference is Hildebrand's book Advanced Calculus for Applications.

Theorem I. **If, on a circular arc C_R with radius R and center at the origin, $z f(z) \to 0$ uniformly as $R \to \infty$, then**

$$\lim_{R\to\infty}\int_{C_R} f(z)\, dz = 0.$$

Theorem II. Suppose that, on a circular arc C_R with radius R and center at the origin, $f(z) \to 0$ uniformly as $R \to \infty$. Then:

1. $$\lim_{R\to\infty} \int_{C_R} e^{imz} f(z)\, dz = 0 \qquad (m > 0)$$

if C_R is in the first and/or second quadrants.*

2. $$\lim_{R\to\infty} \int_{C_R} e^{-imz} f(z)\, dz = 0 \qquad (m > 0)$$

if C_R is in the third and/or fourth quadrants.

3. $$\lim_{R\to\infty} \int_{C_R} e^{mz} f(z)\, dz = 0 \qquad (m > 0)$$

if C_R is in the second and/or third quadrants.

4. $$\lim_{R\to\infty} \int_{C_R} e^{-mz} f(z)\, dz = 0 \qquad (m > 0)$$

if C_R is in the first and/or fourth quadrants.

* This result is known as *Jordan's lemma.*

Theorem III. If, on a circular arc C_ρ with radius ρ and center at $z = a$, $z f(z) \to 0$ uniformly as $\rho \to 0$, then

$$\lim_{\rho\to 0} \int_{C_\rho} f(z)\, dz = 0.$$

Theorem IV. Suppose that $f(z)$ has a simple pole at $z = a$, with residue Res (a). Then, if C_ρ is a circular arc with radius ρ and center at $z = a$, intercepting an angle α at $z = a$, there follows

$$\lim_{\rho\to 0} \int_{C_\rho} f(z)\, dz = \alpha i \text{ Res } (a),$$

where α is positive if the integration is carried out in the counterclockwise direction, and negative otherwise.

NEXT TIME: Return to Green functions briefly and conformal mapping.