LAST TIME: Two-dimensional diffraction – rectangular and circular apertures, Bessel functions, and resolution

REMINDER:

Two-dimensional diffraction

Our diffraction integral now becomes two-dimensional so, proceeding as before, we see the figure to dexcribe this effect as shown to the right. Now

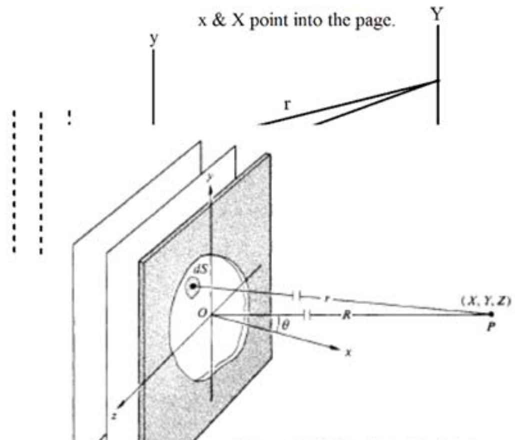


Figure 10.18 Fraunhofer diffraction from an arbitrary aperture, where r and R are very large compared to the size of the hole.

$$r = [X^2 + (Y-y)^2 + (Z-z)^2],$$

so approximately,

$$r = R[1 - (Yy + Zz)/R^2]$$

and

$$E(Y,Z) = \frac{\mathcal{E}_A e^{-ikR}}{R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{A}(y,z)\, e^{ik(yY+zZ)/R}dydz,$$

Make the substitution $\beta' = \frac{kby}{2R}$ and $\alpha' = \frac{kaz}{2R}$ to obtain

$$I(Y,Z) = I(0)\left(\frac{\sin\alpha'}{\alpha'}\right)^2\left(\frac{\sin\beta'}{\beta'}\right)^2.$$

Circular apertures are much more interesting and useful because they are the shape of most lenses.

From before, we obtained

$$E(Y,Z) = \frac{\mathcal{E}_A e^{-ikR}}{R}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}(1)e^{ik(yY+zZ)/R}dydz\,.$$

It makes no sense to try to deal with a circular aperture using Cartesian coordinates, so we change to polar coordinates using the following transformations.

$$z = \rho\cos\phi;\;\; y = \rho\sin\phi\,;\;\; Z = q\cos\Phi\,;\;\; Y = q\sin\Phi$$

The area element in polar coordinates is given by $dS = \rho d\rho d\phi$. Therefore, our integral in polar coordinates is given by

$$E(q,\phi) = \frac{\mathcal{E}_A e^{-ikR}}{R} \int_0^a \int_0^{2\pi} \mathcal{A}(\rho,\phi) e^{i\left(\frac{k\rho q}{R}\right)\cos(\phi-\Phi)} \rho d\rho d\phi.$$

$$E(q,\phi) = \frac{\mathcal{E}_A e^{-ikR}}{R} (2\pi) a^2 \left(\frac{R}{kaq}\right) J_1\left(\frac{kaq}{R}\right)$$

and

$$I(q) = \left(\frac{2\mathcal{E}_A{}^2 A^2}{R^2}\right)\left[\frac{J_1(kaq/R)}{kaq/R}\right]^2.$$

$$I(\theta) = I(0)\left[\frac{2J_1(ka\sin\theta}{ka\sin\theta}\right]^2$$

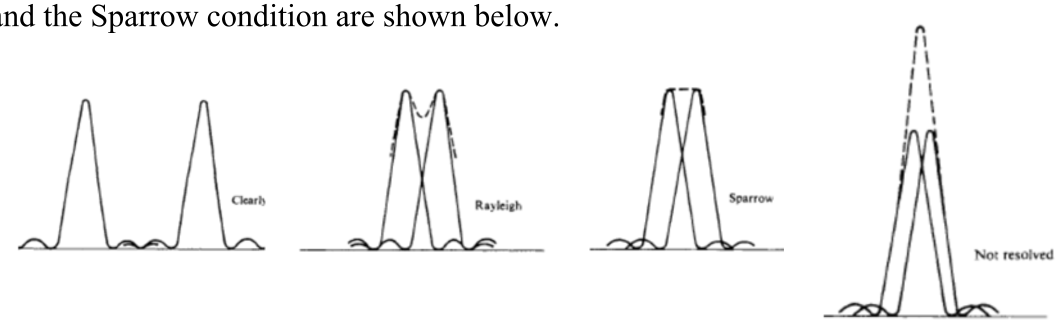
From the table or graph, we wee that the first zero occurs at 3.83, so $\frac{kaq}{R} = \frac{2\pi aq}{\lambda R} = 3.83$ and $q_1 = \frac{\lambda R}{2a}(1.22)$. The very high central peak is know as the Airy disk, named after the person who first derived the expression. On of the most important applications of this result occurs when we try to resolve to objects very close together. If a lens is placed in the aperture, then the value for *R* becomes the focal length *f* of the lens, and 2*a* is the diameter of the lens. This means that the radius of the first bright spot is given by

$$q_1 = 1.22\frac{f\lambda}{D} = \frac{(1.22)(127\text{ mm})633\times10^{-6}\text{mm}}{25.4\text{ mm}} = 4\text{ micrometers}$$

This means that if you filled the lens with the laser beam, the focused spot size would be about 4 micrometers in radius. Unfortunately, however, the unexpanded laser beam does not fill the entire lens, so if the laser beam is onlyl about 1 mm in diameter, then the radius of the central spot is about 100 micrometers. This example shows the advantage of using large lenses and completely filling the aperture with the beam. It also illustrates why the blue-ray dvds are so useful. The shorter the wavelength, the smaller the radius of the central spot size, all other things being equal.

Resolution

The Rayleigh criterion for resolving two objects is to state that the minimum of one of the diffraction patterns just overlaps the maximum of the other object. The figures for this condition and the Sparrow condition are shown below.



Last time we found that $q_1 = \frac{\lambda R}{2a}(1.22)$ and with a lens, $q_1 = 1.22\frac{f\lambda}{D}$.

Therefore, we may also use an angular measure as the resolution limit. Then

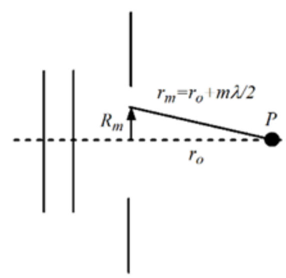
$$\Delta\theta = \frac{q_1}{f} = 1.22\frac{\lambda}{D},$$

so for a wavelength of 550 nm and a lens or mirror diameter D = 3.5 m, which is the diameter of the main adaptive optics telescope at the Starfire Optical Range I mentioned earlier, its diffraction limit of Rayleigh resolution is given by

$$\Delta\theta = \frac{q_1}{f} = 1.22\frac{\lambda}{D} = 1.22\left(\frac{550\times10^{-9}}{3.5}\right) = 1.92\times10^{-7}\text{ rad.}$$

There are 57.2 degrees/rad, and 3600 arc seconds/degree, so this resolution is about 0.04 arc seconds. This telescope has separated binary stars with a separation of 0.1 arc seconds. For an object 100 miles away or about 160 km =1.6 x 10^5 m away, the telescope could resolve points on the object about 0.03 m or 3 cm apart. When optical devices are able to have this type of resolution, we say that it has diffraction limited resolution.

Circular apertures not in the Fraunhofer limit. This are said to be operating in the near-field limit instead of the far-field limit. The easiest case to discuss is the case of a circular aperture with plane wave incident on it, but our observation plane is not in the far-field limit. What this means is that on our observation plane, there is a considerable phase difference between waves arriving from different points in the circular aperture. Here is a cross section view of an on-axis observation point from a circular aperture. For these situations, it turns out to be convenient to break our aperture into regions called Fresnel zones. R_m is the radius of the m^{th} Fresnel zone, and each Fresnel zone boundary marks the change in phase by π which is equivalent to a path difference of $\lambda/2$. Using these conditions, we calculate the radius of the mth Fresnel zone as



$$r_o^2 + R_m^2 = r_m^2 = \left(r_o + \frac{m\lambda}{2}\right)^2 = r_o^2 + mr_o\lambda + \frac{m^2\lambda^2}{4}$$

so

$$R_m \cong \sqrt{mr_o\lambda}\ with\ \frac{m^2\lambda^2}{4} \ll 1.$$

How do we make use of these ideas? We have divided the circular aperture into circular zones (rings) each of which is on the average pi out of phase with the previous zone. We know that the wave across the aperture is a plane wave, so we can calculate the electric field contribution from each zone if we know the area of each zone *A*. Therefore,

$$A = A_{m+1} - A_m = \pi(R_{m+1}^2 - R_m^2) = \pi(m+1)r_o\lambda - \pi mr_o\lambda = \pi r_o\lambda.$$

Let's remove any aperture and see what the irradiance on the screen is when we have an unobstructed wave. Call that value E_u. Then, we obtain

$$E_u = E_1 + E_2 + E_3 + \cdots + E_m$$

$$= |E_1| - |E_2| + |E_3| - |E_4| + \cdots \pm |E_m|$$

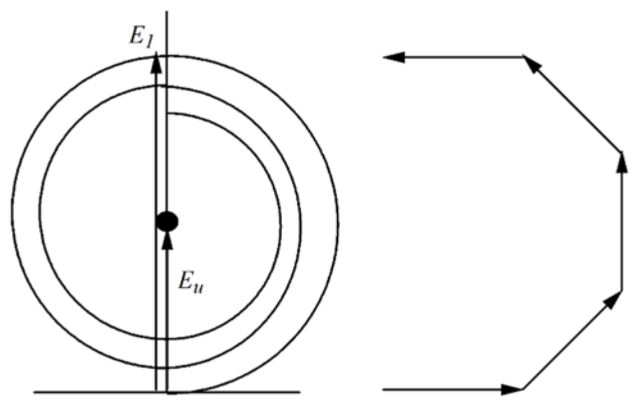
$$= \frac{|E_1|}{2} + \left(\frac{|E_1|}{2} - |E_2| + \frac{|E_3|}{2}\right) + \cdots + \left(\frac{|E_{m-2}|}{2} - |E_{m-1}| + \frac{|E_m|}{2}\right) \pm \frac{|E_m|}{2}.$$

Each of the terms in parenthesis gives approximately zero, so

$$E_u = \frac{|E_1|}{2} \pm \frac{|E_m|}{2}.$$

Once m becomes very large, the obliquity factor begins to reduce its value, so we may neglect the last term in the sum above. Therefore,

$$E_u = \frac{E_1}{2},$$



And this leads to the somewhat surprising and counter-intuitive result that the irradiance with only one zone inside the aperture is given by $E_1 = 2E_u$ or $I_1 = 4I_u$. This means that placing an aperture with only one Fresnel zone gives an irradiance of 4 times the unobstructed wave!!! A rather strange result. Here is a phasor diagram of how this works. The right and left halves of these semicircles represent the odd and even zone contributions, respectively, as m increases. As you can see from this figure, the amplitude of the unobstructed wave is ½ the amplitude of the first zone contribution. The figure to the far right is the discrete version of the continuous curve. Let's see what happens when we have adjusted our parameters so that only 1 zone appears in the aperture. Then

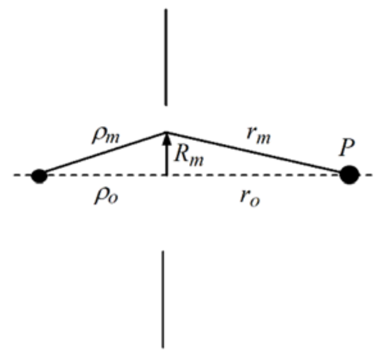
$$R_1 \cong \sqrt{r_o\lambda} = D,$$

where D is the diameter of the aperture. This means that $D^2 = r_o\lambda$, but recall that our condition for Fraunhofer diffraction was given by

$$\lambda \gg \frac{b^2}{R}.$$

Here, however, $b = D$ and $R = r_o$, so what the Fraunhofer condition really means is that the radius of the first Fresnel zone is much larger that the radius of the aperture and only a small fraction of the first zone is inside the aperture. Therefore, the phase change across the aperture is nearly zero, which means the wave is very much like a plane wave, exactly our condition for Fraunhofer diffraction!!! As we move closer to the aperture, we start to see parts of the second zone, so the intensity decreases until when the second zone is in the aperture, the intensity becomes nearly zero. Moving even closer, we bring in the third zone to get another maximum, and so forth. We may also bring in additional zones by leaving the observation distance fixed and increasing the aperture size. Here is a figure from your textbook showing the changes in the pattern as the number of zones increases.

What happens if we have spherical waves incident on an aperture? The basic difference is that the nature of the zones changes. Here is the figure to represent this case.



$$\rho_m^2 = \rho_o^2 + R_m^2 \;\; and \;\; r_m^2 = r_o^2 + R_m^2$$

To get the half-period zones, we use the equation given by

$$(\rho_m + r_m) - (\rho_o + r_o) = \frac{m\lambda}{2}.$$

Now we need to make the approximation that $R_m \ll \rho_o$ and r_o so that we may write

$$\rho_m = \sqrt{(\rho_o^2 + R_m^2)} \cong \rho_o + \frac{R_m^2}{2\rho_o}$$

and

$$r_m = \sqrt{(r_o^2 + R_m^2)} \cong r_o + \frac{R_m^2}{2r_o}.$$

We substitute each of these values into the equation above to obtain

$$\rho_o + \frac{R_m^2}{2\rho_o} + r_o + \frac{R_m^2}{2r_o} - (\rho_o + r_o) = \frac{m\lambda}{2},$$

which implies that

$$\frac{R_m^2}{\rho_o} + \frac{R_m^2}{r_o} = m\lambda \;\; or \;\; \frac{1}{\rho_o} + \frac{1}{r_o} = \frac{m\lambda}{R_m^2}.$$

The physics here is the same, and calculating the area of each zone proceeds in the same way. It turns out that the area of eahc zone is given by

$$A = \left(\frac{\rho_o}{\rho_o + r_o}\right)\pi r_o \lambda.$$

We can obtain the approximate number of zones inside an aperture by divding the area of the aperture by the area of each zone, so the numbe of zones in an aperture is just given by

$$Nzones = \frac{\pi R^2}{\left(\frac{\rho_o}{\rho_o + r_o}\right)\pi r_o \lambda} = \frac{(\rho_o + r_o)R^2}{\rho_o r_o \lambda}.$$

Once again, an even number of zones results in a minimum at the center of the pattern, whereas an odd number of zones results in a maximum irradiance at the center of the pattern. Off-axis irradiances are more complicated, so in the interest of covering other material, I will defer that until later if we get to it.

Comments on barriers.

The geometry for considering a rectangular aperture is the same as for a circular aperture, but you can see that attempting to do the problem in the same way leads to difficulties because our zones do not fit perfectly into rectangular apertures. This will lead us to going back to the diffraction integral and making some approximations that include additional terms that we neglected for Fraunhofer diffraction. This will be our main subject Wednesday.

Videos 30 and 25

NEXT TIME: Fresnel diffraction by rectangular apertures.