LAST TIME: Cauchy integral formula, sequences, series, and Laurent series

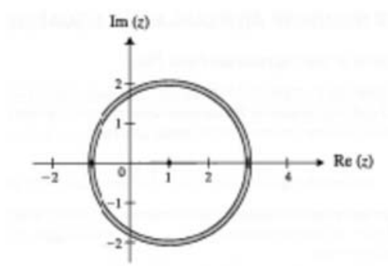
$$f(z) = \sum_{n=-\infty}^{\infty} c_n\,(z-a)^n,$$

where the $c_n{}'s$ are given uniquely by

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^{n+1}}\,d\xi.$$

When $n < 1$, the series contains inverse powers and is said to be the principal part. The positive values for n are the normal Taylor series. To calculate the Laurent series, you may use directly the expressions above, but they do not turn out to be the best method. In many cases, it is possible to rearrange the expression for which you wish to determine the Laurent series in a form that allows you to use the geometric series expansion. Consider the following example.

Example



Determine the series for the function $\frac{1}{z^2-1}$ about the point $z = 1$. We notice that the function is not analytic at the two points $z = \pm 1$. The figure shows that we may find a Laurent series for each of the two regions given by

$$0 < |z-1| < 2 \text{ and } |z-1| > 2.$$

Therefore $z = -1$ is outside the annulus, and $z = 1$ is inside an infinitesimal hole at the center. Consider the case inside first. Let's rewrite the function in a different way by manipulating it a bit to obtain

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

We want to make the second term have a term $(z-1)$ in it, so we write it as

$$\frac{1}{z+1} = \frac{1}{z-1+2} = \frac{1}{2\left(1+\frac{z-1}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n.$$

Now combine the first term with this result to yield

$$\frac{1}{z^2-1} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n\right]$$

$$= \frac{1}{4} \sum_{n=-1}^{\infty} (-1)^{n+1} \left(\frac{z-1}{2}\right)^n ,$$

Where we incorporated the $\frac{1}{z-1}$ into the expression and changed the limits of the sum to reflect only one negative power of $z-1$. The Laurent series in this region has only one negative power of $z-1$. We will see the importance of this fact later.

Now we look at the annular region for $|z-1| > 2$. This time, we expand using the inverse to obtain

$$\frac{1}{z+1} = \frac{1}{z-1+2} = \frac{1}{(z-1)\left(1+\frac{2}{z-1}\right)} = \frac{1}{z-1} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z-1}\right)^n .$$

Again, we insert the first term to obtain

$$\frac{1}{z^2-1} = \frac{1}{2}\left[\frac{1}{z-1} - \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^n}{(z-1)^{n+1}}\right)\right]$$

$$= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-2)^n}{(z-1)^{n+1}} .$$

You will notice that this series has infinitely many negative powers of $z-1$ and no positive powers.

Zeros and Singularities

The point $z=a$ is a zero of a function $f(z)$ if $f(a) = 0$. If $f(z)$ is analytic at a, we may expand it in a Taylor series in some region where $|z-a| < \rho$, so

$$f(z) = \sum_{n=0}^{\infty} c_n \, |z-a|^n .$$

For a being a zero of $f(z), c_o = 0$. However, if $c_1 \neq 0$, then a is called a simple zero of $f(z)$. If both c_o and c_1 are zero, but $c_2 \neq 0, a$ is a zero of order two. As a general statement, if all coefficients out to c_n are zero, but c_n is not zero, the zero is of order n. What this is really telling us is how fast the function approaches zero as z approaches a. A rather easy example is the function $\sin z^2$, whose Taylor series is given by

$$z^2 - \frac{z^6}{3!} + \cdots$$

Here, we see that c_o and c_1 are zero, but $c_2 \neq 0$, so $z=0$ is a second order zero.

It is possible that a function $f(z)$ may be analytic in the neighborhood of a point $z = a$, but not analytic at a. We call point a an isolated singularity of $f(z)$. Three cases are considered below.

1. Suppose $|f(z)| \to \infty$ as $z \to a$. These singularities are called poles. The simplest example is just

$$f(z) = \frac{1}{z - a}.$$

2. $f(z)$ is bounded as $z \to a$. Consider the function

$$f(z) = \frac{\cos z}{z - \frac{\pi}{2}}.$$

Here, $\frac{\pi}{2}$ is an isolated singularity, but we may use l'Hospital's rule to determine that the limit is given by

$$\lim_{z \to \frac{\pi}{2}} \frac{\cos z}{z - \frac{\pi}{2}} = \lim_{z \to \frac{\pi}{2}} \frac{-\sin z}{1} = -1.$$

For these cases, we may redefine the function as

$$f(z) = \begin{cases} \dfrac{\cos z}{z - \frac{\pi}{2}} & z \neq \dfrac{\pi}{2} \\ -1 & z = \dfrac{\pi}{2} \end{cases}$$

and no problems exist.

3. $f(z)$ may oscillate.

The singularities of the first type are especially important, so we discuss them now. These are called poles. The function is analytic in an annular region given by $0 < |z - a| < \rho$. The Laurent series is given by

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \, (z - a)^n.$$

If the coefficient $c_{-1} \neq 0$, but all $c_{-m} = 0$ for $m > 1$, then the pole is of order one and is said to be a simple pole. If $c_{-m} \neq 0$, the pole is order m. There is a way to test for the order of a pole without having to find the Laurent series. Consider

$$\lim_{z \to a} (z - a)^p \, f(z) = \lim_{z \to a} (z - a)^p \sum_{n=-\infty}^{\infty} c_n \, (z - a)^n$$

$$= \lim_{z\to a} \sum_{n=-\infty}^{\infty} c_n\,(z-a)^{n+p}.$$

$\lim_{z\to a} \sum_{n=-\infty}^{\infty} c_n\,(z-a)^{n+p} = 0$ for $p > m$, will be constant c_{-m} for $p = m$, and does not exist for $p < m$. We conclude that the order of a pole a z = a is the lowest integer p for which

$$\lim_{z\to a}(z-a)^p\,f(z).$$

exists.

The Residue Theorem

Suppose $f(z)$ is analytic in a neighborhood of $z = a$ except perhaps at $z = a$. The residue of $f(z)$ at a is given by

$$res\ f\ at\ a = \frac{1}{2\pi i}\oint_C f(z)dz,$$

where C encloses a. This means that the residue = zero if $f(z)$ is analytic at a. Note, however, that a zero residue does not imply that the function is analytic at a. Recall that the value of c_n in the Laurent expansion is given by

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^{n+1}}\,d\xi,$$

so when we compare the residue theorem with the Laurent expansion, we see that the value of c_{-1} is the residue at a. If a function $f(z)$ in a simply connected domain is analytic except for a finite number of isolated singularities and if the curve C is within the domain, then

$$\oint_C f(z)dz = 2\pi i \sum_{n=1}^{N} Res\ f(z_n) = 2\pi i \sum_{n=1}^{N} K_n,$$

where z_n are the singularities of $f(z)$ within C. This is a very powerful theorem, and we want to give numerous examples of its use. I have used K_n for the residues. There are numerous methods of calculating these residues, and I list them below.

1. We may calculate the Laurent series and pick out the coefficient c_{-1}.

2. For a simple pole, the residue at a is given by

$$Res\ f(a) = K_n(a) = \lim_{z\to a}(z-a)\,f(z).$$

Number 2 is true because if you substitute the Laurent series in for $f(z)$, it yields only c_{-1}.

3. For a pole of order m, we may calculate the residue by considering the Laurent series once again and differentiate it term by term, and we obtain

$$Res\, f(a) = K_n(a) = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)].$$

If you want to calculate the residues by the Laurent series, it is useful to remember the following series summation given by

$$\frac{1}{1-z} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} z^n, & |z| < 1 \\ \displaystyle -\sum_{n=1}^{\infty} \frac{1}{z^n}, & |z| > 1 \end{cases}.$$

Determine the residues for the function given by

$$\frac{z^2}{(z-2)(z^2+1)}.$$

We first find the poles and determine their order. Here, it is clear that the simple poles are $z = 2, i, -i,$ so we may use method 2 listed above to find the residues. Therefore,

$$K(2) = \lim_{z\to 2} (z-2) \left\{ \frac{z^2}{(z-2)(z^2+1)} \right\} = \frac{2^2}{2^2+1} = \frac{4}{5},$$

$$K(i) = \lim_{z\to i} (z-i) \left\{ \frac{z^2}{(z-2)(z+i)(z-i)} \right\} = \frac{i^2}{(i-2)(i+i)} = \frac{-1}{2i(i-2)} = \frac{1-2i}{10},$$

and

$$K(-i) = \lim_{z\to i} (z+i) \left\{ \frac{z^2}{(z-2)(z+i)(z-i)} \right\} = \frac{(-i)^2}{(-i-2)(-i-i)} = \frac{-1}{-2i(-i-2)} = \frac{1+2i}{10}$$

Just to illustrate, you may also use method 3 even though all poles are simple poles and do not require it. As one example, consider $K(i)$. Use m = 1 to obtain

$$K(i) = \lim_{z\to i} \frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}} \left[(z-i)^1 \frac{z^2}{(z-2)(z+i)(z-i)} \right] = \frac{i^2}{(i-2)(i+i)},$$

and the result is the same as before. Here is one that is a slightly more challenging. Consider

$$f(z) = \frac{ze^{zt}}{(z-3)^2}.$$

Here, $z = 3$ is a pole of order 2, so we use method 3 to obtain

$$K(3) = \lim_{z\to 3} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left[(z-3)^2 \frac{ze^{zt}}{(z-3)^2} \right] = \lim_{z\to 3} \frac{d}{dz} (ze^{zt}) = \lim_{z\to 3} (e^{zt} + zte^{zt})$$

$$= e^{3t} + 3te^{3t}.$$

Now let's move to the combination of finding the residues and using them to calculate an integral. Evaluate the integral

$$\oint_C \frac{e^z dz}{(z-1)(z+3)^2}$$

where C is given by (a) $|z| = \frac{3}{2}$ and (b) $|z| = 10$. We notice that there is a simple pole at $z = 1$ and a double pole (order 2, m = 2) at $z = -3$.

We calculate the residues to obtain

$$K(1) = \lim_{z\to 1}(z-1)\left\{\frac{e^z}{(z-1)(z+3)^2}\right\} = \frac{e}{16}$$

and for $z = -3$, we obtain

$$K(-3) = \lim_{z\to -3}\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z+3)^2\frac{e^z}{(z-1)(z+3)^2}\right] = \lim_{z\to -3}\frac{d}{dz}\frac{e^z}{(z-1)}$$

$$= \lim_{z\to -3}\frac{(z-1)e^z - e^z}{(z-1)^2} = \frac{-5e^{-3}}{16}.$$

For $|z| = \frac{3}{2}$, only the pole at $z = 1$ is enclosed, so the value for the integral is given by

$$2\pi i\frac{e}{16} = \frac{\pi i e}{8}.$$

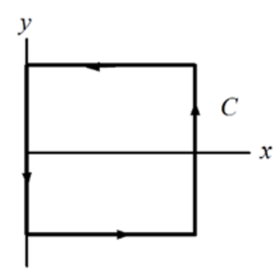
For $|z| = 10$, both poles are included, so the value for the integral is given by

$$\frac{\pi i e}{8} - 2\pi i\frac{-5e^{-3}}{16} = \frac{\pi i(e - 5e^{-3})}{8}.$$

Let's take a quick look at a different contour by evaluating the integral given by

$$\oint_C \frac{\sin z}{z-1}dz,$$

Where C is given by a square centered at $z = 1$. The corners are located at the points given by $-i, 2-i, 2+i, i$. The figure shows the contour. There is a simple pole at $z = 1$, and it is inside the contour. Therefore, The residue is given by



$$K(1) = \lim_{z\to 1}(z-1)\frac{\sin z}{z-1} = \sin 1$$

and

$$\oint_C \frac{\sin z}{z-1}dz = 2\pi i \sin 1 = 5.29i.$$

One use of the residue theorem is to calculate the integral of functions having the form

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta.$$

To see how this works with complex integration, we need to convert the sine function, cosine functions, and $d\theta$ to their complex forms. We use $z = re^{i\theta}, dz = ire^{i\theta}d\theta = ie^{i\theta}d\theta$ if we evaluate the integrals around the unit circle in the complex plane. Therefore, $d\theta = \frac{dz}{iz}$.

We express sine and cosine in terms of *z* as follows.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Using these transformations, we may convert trigonometric functions to functions of *z* and use contour integration to obtain useful results.

NEXT TIME: Examples and other applications of contour integration