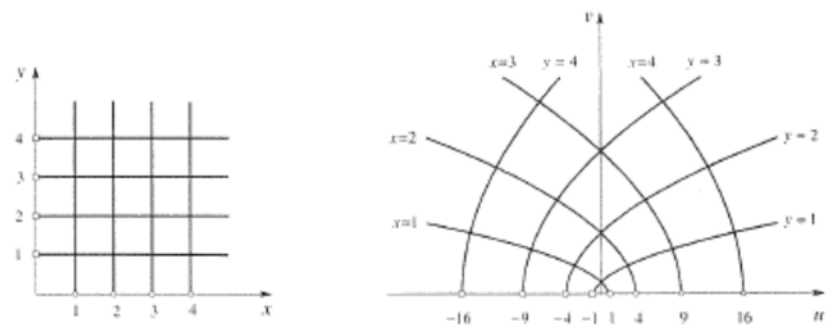
LAST TIME: Complex elementary functions, mapping, differentiability, and Cauchy theorem

Before we leave the mapping discussion, it is worthwhile noting that the mapping need not translate, stretch, rotate, or invert, you might just want to see how a particular function $f(z)$ appears in the $u-v$ plane. Let's look at how $w=f(z)=z^2$ appears in the $u-v$ plane when we consider a particular set of curves in the $x-y$ plane. We look at a set of curves $x=1, x=2, x=3, x=4, y=1, y=2, y=3, and\ y=4$. Just to see how one looks, consider $x=1$, where y extends from 0 to ∞. For this function, $u(x,y)=x^2-y^2$ and $v(x,y)=2xy$. Therefore, $u(x,y)=1-y^2\ and\ v(x,y)=2y$. So, $u=1-\frac{v^2}{4}$. Similar shapes occur for the other straight lines. Here are the graphs of each set of curves in the $x-y$ plane and in the $u-v$ plane.



Analyticity

A function that is differentiable at $z=a$ and within a neighborhood of $z=a$ is analytic at $z=a$. Since a complex function represents a mapping of the z-plane onto the w-plane, any integral given by

$$\int_{z_1}^{z_2} f(z)dz$$

is an integral along a path that must be specified between the points z_1 and z_2. We may rewrite this integral by expanding the integrand and the differential to obtain

$$\int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2} (u+iv)(dx+idy) = \int_{z_1}^{z_2} [u\,dx - v\,dy + i(u\,dy + v\,dx)].$$

We define the vectors $\mathbf{u}$ and $\mathbf{v}$ by $\mathbf{u}=(\mathrm{u},-\mathrm{v})$ and $\mathbf{v}=(\mathrm{v},\mathrm{u})$, we may rewrite the integral as

$$\int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2} \mathbf{u}\cdot \mathrm{d}\boldsymbol{\ell} + i\int_{z_1}^{z_2} \mathbf{v}\cdot \mathrm{d}\boldsymbol{\ell}$$

We are already familiar with line integrals from our previous work. We want to consider evaluating this integral along a path from z_1 to z_2 along C_1 and back to z_1 along $C_2 -$ in other words, along a closed path. Therefore, we obtain

$$\oint f(z)dz = \oint \mathbf{u} \cdot \mathrm{d}\boldsymbol{\ell} + i \oint \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell},$$

Where all integrals are along $C = C_1 + C_2$. Sometime ago, we mentioned Green's theorem in the plane, which is given by

$$\oint (f\ dx + g\ dy) = \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy,$$

which, as I hope you remember, is just Stokes' theorem for the special case where the curve lies entirely in the *x*-*y* plane. We obtain, by assigning the proper values to f and g

$$\oint f(z)dz = \iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\ dy + i \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\ dy = 0 \text{ by the Cauchy} - \text{Riemann}$$

relations. A fundamental theorem states that if $f(z)$ is analytic in and on C

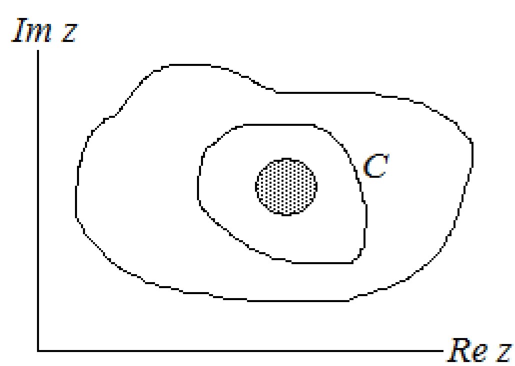
$$\oint f(z)dz = 0.$$

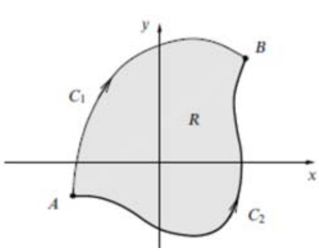
This is Cauchy's theorem. We will find many uses for it during our studies of complex integration.

To continue with the integral development, we wish to consider some of the interesting of Cauchy's theorem. We begin by looking at the integral I given by

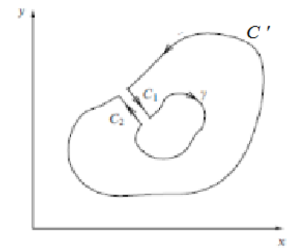
$$I = \oint_C \frac{1}{z-a} dz,$$

where C is the closed curve in the complex plane. The function $f(z) = \frac{1}{z-a}$ is everywhere analytic except at $z = a$. If we take a curve that does not include $z = a,$ then the value of the integral should be zero. The curve shown here is not simply connected; i.e., we cannot shrink C to a point and stay in the region. One interesting bit of information concerns the behavior of the contours from *A* to *B* as shown. You may see that so the values of the two integrals are the same.





On the other hand, we may construct a curve that does not include the point, which is inside the shaded region, as shown in the figure. The curve shown in this figure does not include point a. Therefore,



$$I = \oint_{C'} \frac{1}{z-a} dz = \oint_{C+C_1+C_2+\gamma} \frac{1}{z-a} dz$$

$$= \oint_{C} \frac{1}{z-a} dz + \oint_{\gamma,cw} \frac{1}{z-a} dz,$$

where the values along $C_1$and C_2 cancel because the curves are constructed to have equal length but are traveled in opposite directions. Remember that the usual convention is to travel the curve in such a way that point a is to your left (ccw). From these results, we see that

$$\oint_{C} \frac{1}{z-a} dz = \oint_{\gamma} \frac{1}{z-a} dz.$$

On the curve γ, let's write $z = a + \rho e^{i\theta}$, so that $dz = i\rho e^{i\theta} d\theta$ and

$$\oint_{C} \frac{1}{z-a} dz = \oint_{\gamma} \frac{1}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta = i \int_0^{2\pi} d\theta = 2\pi i.$$

The conclusion from all this is that

$$\oint_{C} \frac{1}{z-a} dz = \begin{cases} 2\pi i & if\ z = a \text{ lies within } C \\ 0 & \text{otherwise} \end{cases}$$

This is known as Cauchy's integral formula. We want to generalize this in a couple of ways for future use. Consider

$$\oint_{C} \frac{f(z)}{z-a} dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta$$

and take the limit as $\rho \to 0$ to obtain

$$\oint_{C} \frac{f(z)}{z-a} dz = \begin{cases} 2\pi i f(a) & if\ z = a \text{ lies within } C \\ 0 & \text{otherwise} \end{cases}$$

Finally, we may get the most general Cauchy formula given by

$$\oint_{C} \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z)|_{z=a}.$$

This is also sometimes written as

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz.$$

What do we expect to gain from these formulas and ideas? There are numerous applications where we may encounter expressions where the denominator has a form of $(z-a)$ or multiple occurrences of such values. We need a way to evaluate these in many physical situations. Consider the following example. Evaluate

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz.$$

where $C$is the circle $|z| = 3$. Each of the points $z = 1$ and $z = 2$ lie within the circle. The function $f(z)$ is given by $\sin \pi z^2 + \cos \pi z^2$. This is an ideal case for the Cauchy integral theorem. We need to apply the theorem for each of the two values in the denominator by using the idea of partial fractions. Therefore,

$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

and

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz.$$

Finally,

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz = 2\pi i[\sin 4\pi + \cos 4\pi] = 2\pi i$$

and

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz = 2\pi i[\sin \pi + \cos \pi] = -2\pi i.$$

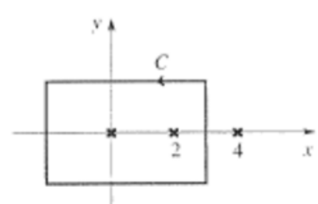
Then

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i - -2\pi i = 4\pi i.$$

Let's look at one more that is a bit different. Evaluate

$$I = \oint_C \frac{1}{z^2(z-2)(z-4)} dz,$$

where the curve is shown to the right. What we notice in this case is that the point $(z = 4)$ does not lie within the curve. This means that it will not contribute to the integral according to the Cauchy integral theorem. Once again, we need to put this integral in the standard form by using a partial fraction decomposition.



$$\frac{1}{z^2(z-2)(z-4)} = \frac{3}{32z} + \frac{1}{8z^2} - \frac{1}{8(z-2)} + \frac{1}{32(z-4)}.$$

Because the last term does not contribute, we don't need to keep it in, so we obtain, using $f(z) = 1$

$$I = \frac{3}{32}\oint_C \frac{dz}{z} + \frac{1}{8}\oint_C \frac{dz}{(z-0)^2} - \frac{1}{8}\oint_C \frac{dz}{(z-2)} = \frac{3}{32}(2\pi i) + \frac{1}{8}(0) - \frac{1}{8}(2\pi i) = -\frac{\pi i}{16}.$$

Do you see why the second term is zero? Do we always end up doing our complex integrals over a specified path? In order not to have to use some path, we must state the fundamental theorem of complex integrals. It is the same as the fundamental theorem of real analysis. Specifically,

$$\int_{z_o}^{z} f(z')dz' = F(z) - F(z_o),$$

provided $f(z')$ is analytic in the domain and $F'(z) = f(z)$. For these types of functions, we may just use ordinary limits of integration as in real variable calculus. As an example consider

$$\int_{2i}^{3} \sin z \, dz = -\cos z|_{2i}^{3} = -[\cos 3 - \cos 2i] = \cosh 2 - \cos 3.$$

Remember that sine, cosine, exponentials, and polynomials are always analytic everywhere, so we may expect that no contour will need to be specified to do their integrals.

A brief review of sequences and series is in order for what follows. I will simply state a few things about real sequences and series for review.

A sequence is an ordered set of numbers a_n with a rule for computing the n^{th} element. a_n is just the n^{th} number in the sequence. A sequence is convergent if a number s exists such that

$|a_n - s| < \epsilon$ so that $\lim_{n\to\infty} a_n = s$. A series is formed by summing the terms of a sequence. If

$$S_1 = \{1,2,3,4,\cdots\},$$

then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n.$$

A second sequence may be formed from the partial sums of the series to yield

$$A_m = \sum_{n=1}^{m} a_n.$$

If this sequence of partial sums converges, then the series converges. One requirement for convergence, is for the successive terms to approach zero; *i.e.*, $a_n \to 0$. If, on the other hand,

$$\lim_{n\to\infty} a_n \neq 0,$$

the series diverges. Here is a review of the basic tests for convergence.

Root test: If

$$\lim_{k\to\infty} |a_k|^{\frac{1}{k}} = q \; and \; q < 1,$$

the series converges absolutely, but if $q > 1$, the series diverges. Otherwise, the test fails.

Ratio test: If

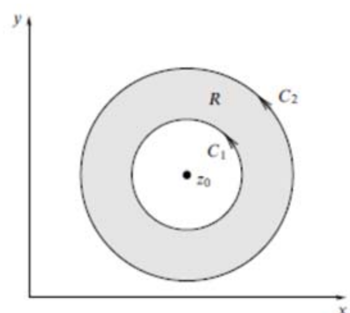
$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = q \text{ with } q < 1,$$

the series converges absolutely. If $q > 1$, the series diverges, and if $q = 1$, the test fails. I will leave it to you to review the integral test, the comparison test, and the alternating series test. Many of the standard tests are listed in your textbook.

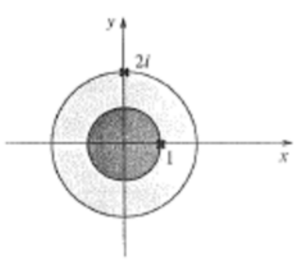
For complex cases, both the real and imaginary parts of the series must be examined, and both must converge for the complex series to converge. Taylor series for complex numbers takes essentially the same form as it does for real numbers. Therefore,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2} f''(a) + \cdots + \frac{(z-a)^n}{n!}\frac{d^n f}{dz^n} +$$

Taylor series uniformly converge within the circle $|z-a| \leq \rho$, where ρ is the radius of convergence. If our function is not analytic within the region, but instead is analytic in an annular region such that $\rho_1 < |z-a| < \rho_2$ (see the figure to the right), then the series will have both positive and negative powers. This series is called a Laurent series. It comes about when we consider a function of the following type given by



$$f(z) = \frac{1}{(z-1)(z-2i)}.$$



You may expand this function in a Taylor's series provided you stay within $|z| < 1$. If, however, you go outside this region, the function is not analytic unless you stay within the annular region or go outside the annular region. The basic theorem here is the following. If a function is analytic within and including the concentric circles bounded by the concentric circles C_1 and C_2, then the function may be expanded in a Laurent series given by

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \,(z-a)^n,$$

where the $c_n{}'s$ are given uniquely by

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi.$$

We may then have three possible Laurent expansions, one for $2 < |z| < \infty$, one for $1 < |z| < 2$, and one for $|z| < 1$, which, of course, is just a Taylor series expansion. The proof of both the Taylor series and the Laurent series is completed by use of the Cauchy formula.

NEXT TIME: Examination 2