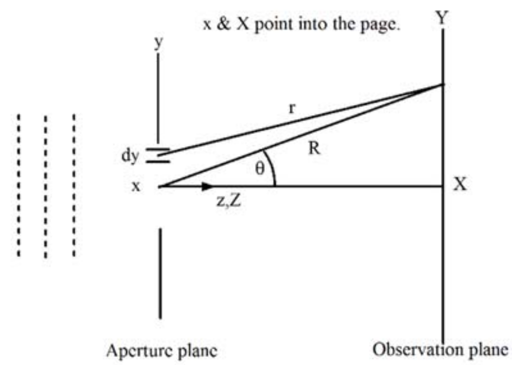
Lecture 19 was examination 2. Return examination and comment.

LAST TIME: Interferometers and multiple-wave interference



To understand interference by wavefront division, we must first take up diffraction because diffraction is present in all wavefront division problems. In this course, we will adopt the Huygens-Fresnel principle as our starting point. Recall that this principle states that waves are generatged by considering the superposition of the previous wavefont with phase considerations taken into account. Two cases of diffraction are usually considered: Fraunhofer (far field where plane waves are assumed) and Fresnel (near field where the curvature of the fields are considered). Because Fraunhofer is easier, it is better to discuss it first. We start with a one-dimensional case first. The figure shows a typical situation. A plane wave is incident on an aperture and each differential element dy acts as the source of a spherical wave and is added to every other differentail element with the phase between them considered. Mathematically, this predicts an equation given by

$$E(Y) = \int \mathcal{E}_L \mathcal{A}(y) \frac{e^{-ikr}}{r} dy,$$

where $\mathcal{E}_L$ is the source strength per unit length and $\mathcal{A}(y)$ is the aperture function. Mention obiquity factor. Note that we have suppressed the time dependence because it is common to all such waves. Comment on why this is done in this way. To use this equation in an analytic way, we have to do several things. First, write r in terms of Z, Y, and y. $r = [Z^2 + (Y - y)^2]^{1/2}$. If you substitute this value directly into the equation above, you will find that the integral cannot be done analytically. Expand, simplify, and approximate to obtain

$$r = [y^2 + Z^2 - 2yY + Y^2]^{1/2} = [R^2 - 2yY + y^2]^{1/2}$$

and

$$r = R\left[1 + \left(\frac{-2yY + y^2}{R^2}\right)\right]^{1/2}$$

so

$$r \cong R\left[1 + \frac{1}{2}\left(\frac{-2yY + y^2}{R^2}\right)\right] \cong R - \frac{yY}{R} + (\text{order } y^2) = R - y\sin\theta$$

Finally,

$$E(Y) = \frac{\mathcal{E}_L e^{-ikR}}{R} \int_{-\infty}^{\infty} \mathcal{A}(y)\, e^{ikyY/R} dy.$$

What does neglecting the quadratic term mean? That term must be much less than one.

Physically, we can look at the term in more detail to understand it better. Consider b to be the slit width.

$$kR\left(\frac{1}{2}\right)\left(\frac{y}{R}\right)^2 \ll 1 \Rightarrow \frac{2\pi}{\lambda}R\left(\frac{1}{2}\right)\left(\frac{b}{R}\right)^2 \ll 1 \Rightarrow \frac{\lambda R}{b^2} \gg 1 \Rightarrow \lambda \gg \frac{b^2}{R}.$$

Comments.

You will notice that I have written the diffraction integral somewhat differently from your book and some what differently from most books. The reason will become obvious when we come to our last part of the course concernign Fourier optics.

Single slit diffraction example

The first thing we need to do is to define the aperture function because that will define our limits of integration and tell us what the integral looks like. For now, we will assume that our aperture function is a completely open aperture with an amplitude of 1 and extends from $-b/2$ to $+b/2$. Therefore, our integral becomes

$$E(Y) = \frac{\mathcal{E}_L e^{-ikR}}{R}\int_{-b/2}^{b/2}(1)\, e^{ikyY/R}dy$$

$$= \frac{\mathcal{E}_L e^{-ikR}}{R}\frac{R}{ikY}\left[e^{ikbY/2R} - e^{-ikbY/2R}\right]$$

$$= \left(\frac{\mathcal{E}_L e^{-ikR}}{R}\right)\left(2i\frac{R}{ikY}\right)\left(\sin\frac{kbY}{2R}\right).$$

$$= \left(\frac{\mathcal{E}_L b e^{-ikR}}{R}\right)\left(\frac{\sin\beta}{\beta}\right) \text{ with } \beta = \frac{kbY}{2R} = \frac{kb\sin\theta}{2}$$

$$I(\theta) = \left(\frac{1}{2}\right)Re(E^*E) = \left(\frac{1}{2}\right)\left(\frac{\mathcal{E}_L b}{R}\right)^2\left(\frac{\sin\beta}{\beta}\right)^2 = I(0)\left(\frac{\sin\beta}{\beta}\right)^2$$

Minima in the diffraction pattern occur whenever $\beta = m\pi, m \neq 0$. Therefore, $\frac{kb\sin\theta}{2} = m\pi$ so $b\sin\theta = m\lambda, m \neq 0$ is the condition for each minimum. To get the maxima, we must calculate the dreivative, set it equal to zero, and solve the equation. This gives the result that

$$\frac{dI}{d\beta} = 0 \Rightarrow \tan\beta = \beta.$$

This is a transcendental equation and is solved numerically to obtain values given by

$$\beta = \pm 1.43\pi, \pm 2.45\pi, \pm 3.47\pi$$

so that

$$\beta \cong \left(m + \frac{1}{2}\right)\pi \text{ with } m = 1,2,\cdots$$

To see two-wave interference from wavefront division, we need another aperture, so we create a double slit with the following properties.

$$\mathcal{A}(y) = 1 \text{ from } -\frac{a}{2} - \frac{b}{2} \le y \le -\frac{a}{2} + \frac{b}{2}$$

$$= 1 \text{ from } +\frac{a}{2} - \frac{b}{2} \le y \le +\frac{a}{2} + \frac{b}{2}$$

$$= 0 \text{ otherwise}$$

where a is the center-to-center slit separation.

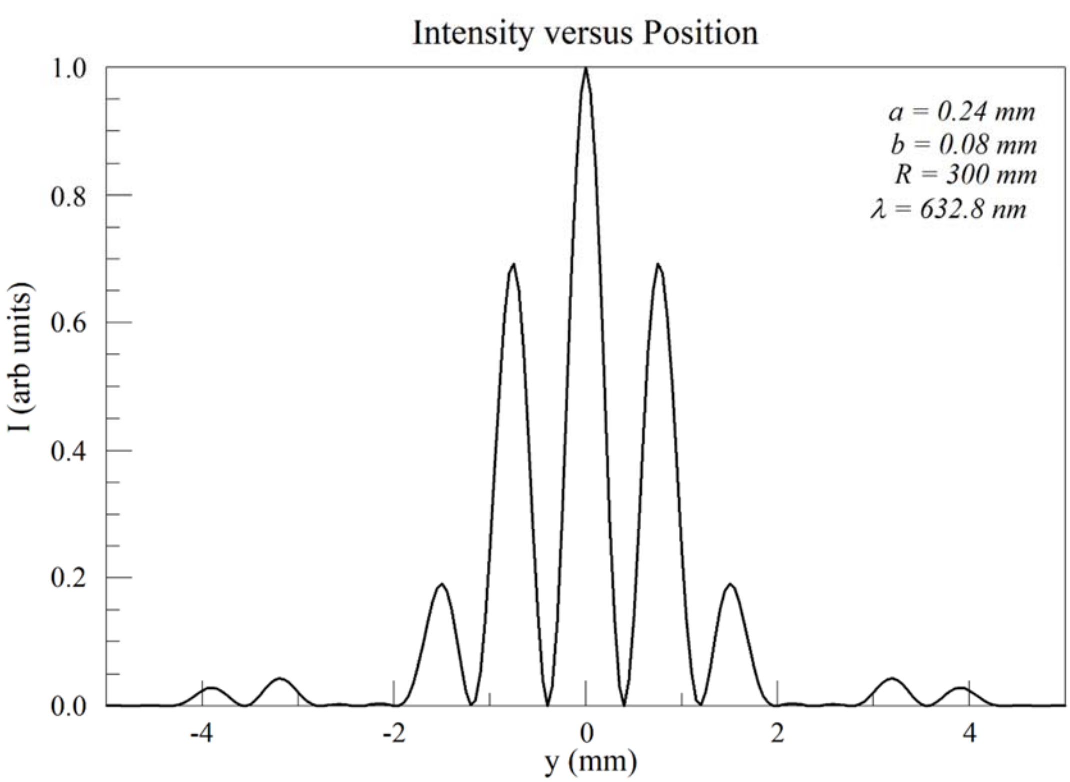
Now the integral becomes (homework problem)

$$I(\theta) = 4I_o\left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha \text{ with } \alpha = \frac{ka\sin\theta}{2}.$$

Therefore,

$$I(\theta) = 0 \text{ for } \beta = \pm m\pi \ \ (m \neq 0) \text{ or } \alpha = \pm\left(m + \frac{1}{2}\right)\pi.$$

The first condition represents the diffraction minimum, and the second condition is the interference condition. Obviously, because the factors are multiplied, the result is zero if either of the conditions is satisfied. Here is a graph showing the results of interference and diffraction for the case of *b* =0.08 mm and *a* = 0.24 mm = 3*b*.



To do the multiple-wave interference problem with apertures, we just need to add in more apertures and see if we can determine a pattern as we did with the multiple-wave interference by reflection. To make our life easier, we put the center of the lower slit at the origin, so our aperture function has the form

$$\mathcal{A}(y) = 1 \text{ for } -\frac{b}{2} \leq y \leq +\frac{b}{2}$$
$$= 1 \text{ for } a-\frac{b}{2} \leq y \leq a+\frac{b}{2}$$
$$= 1 \text{ for } 2a-\frac{b}{2} \leq y \leq 2a+\frac{b}{2}$$
$$= 1 \text{ for } (\mathrm{N}-1)a-\frac{b}{2} \leq y \leq (\mathrm{N}-1)a+\frac{b}{2}$$

Consider the contribution from the j^{th} slit and then do the summation over all slits.

$$E_j = \frac{\mathcal{E}_L e^{-ikR}}{R}\int_{ja-b/2}^{ja+b/2}(1)\, e^{iky\sin\theta}dy$$

$$= \frac{\mathcal{E}_L e^{-ikR}}{ik\sin\theta\, R}\left[e^{ik\sin\theta(ja+b/2)} - e^{ik\sin\theta(ja-b/2)}\right]$$

$$= \frac{\mathcal{E}_L\left(\frac{b}{2}\right)}{ik\sin\theta\left(\frac{b}{2}\right)}\frac{e^{-ikR}}{R}\left[e^{ik\sin\theta(ja)}\left(2i\sin\left(\frac{kb}{2}\sin\theta\right)\right)\right]$$

$$= \mathcal{E}_L b\frac{e^{-ikR}}{R}\left[e^{ikja\sin\theta}\left(\frac{\sin\beta}{\beta}\right)\right]$$

Finally,

$$E = \sum_{j=0}^{N-1} E_j = \mathcal{E}_L b\left(\frac{\sin\beta}{\beta}\right)\frac{e^{-ikR}}{R}\sum_{j=0}^{N-1}\left(e^{i2\alpha}\right)^j$$

We expand the summation to identify it and obtain

$$E = \sum_{j=0}^{N-1} E_j = \mathcal{E}_L b\left(\frac{\sin\beta}{\beta}\right)\frac{e^{-ikR}}{R}\left[1 + e^{i2\alpha} + \left(e^{i2\alpha}\right)^2 + \cdots + \left(e^{i2\alpha}\right)^{(N-1)}\right]$$

The bracket term looks like $S = 1 + x + x^2 + \cdots + x^{(N-1)}$ so we multiply by x and obtain

$$Sx = x + x^2 + \cdots + x^N.$$
$$S - Sx = 1 + x + x^2 + \cdots + x^{(N-1)} - (x + x^2 + \cdots + x^N) = 1 - x^N$$

Therefore,

$$S = \frac{1-x^N}{1-x}$$

and

$$E = \mathcal{E}_L b\left(\frac{\sin\beta}{\beta}\right)\frac{e^{-ikR}}{R}\left[\frac{e^{i2\alpha N}-1}{e^{i2\alpha}-1}\right].$$

We make things look a bit nicer by

$$E = \mathcal{E}_L b\left(\frac{\sin\beta}{\beta}\right)\frac{e^{-ikR}}{R}\frac{e^{i\alpha N}}{e^{i\alpha}}\left[\frac{e^{i\alpha N}-e^{-i\alpha N}}{e^{i\alpha}-e^{-i\alpha}}\right]$$

Now get irradiance given by

$$I = \frac{1}{2}Re(E^{*}E) = I_o\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2,$$

where $I_o =$ one slit so $I(0) = N^2 I_o$. Comment on $\beta \to 0$ case

You can see that we cannot get rid of the diffraction pattern that modulates the multiple-wave interference coming from the second term. Here is a figure from your textbook on the next page.

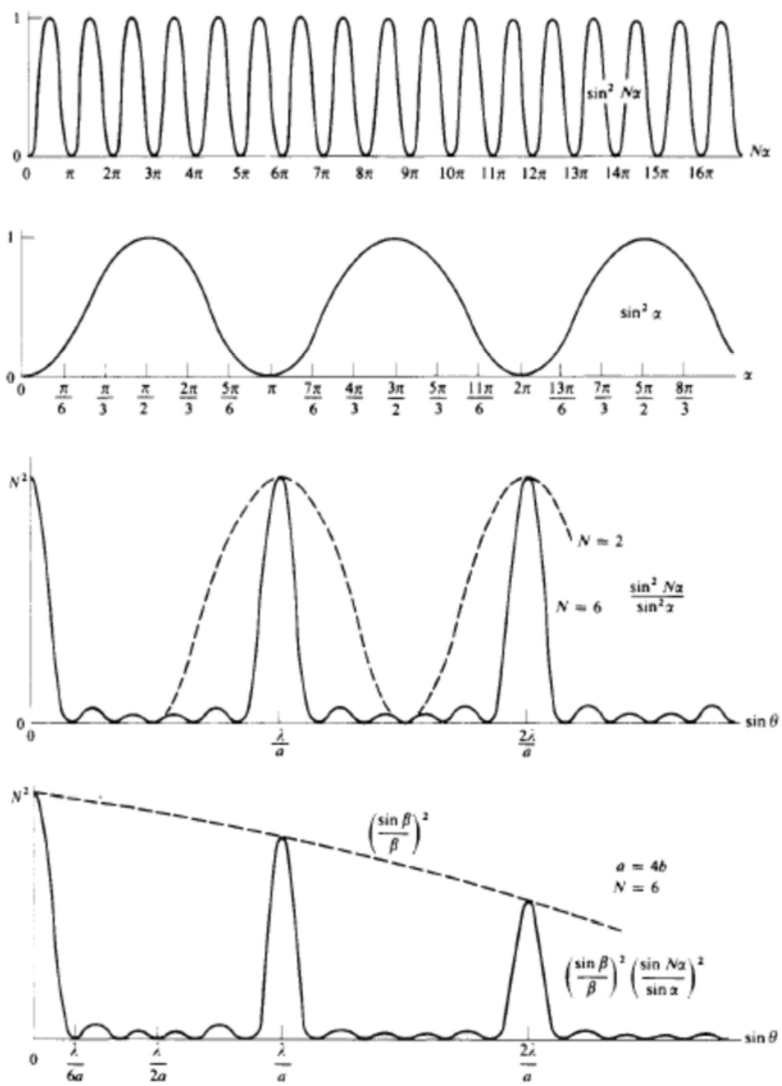


Figure 10.17 Multiple-slit pattern (a = 4b, N = 6).

Principal maxima occur when $\alpha = 0, \pm\pi, \pm 2\pi, etc.$ The diffraction zeros will always be determined by the slit width regardless of how many slits there are. Minima in the irradiance occur when $\left(\frac{\sin N\alpha}{\sin \alpha}\right) = 0$ or at values of $\alpha = \pm\frac{\pi}{N}, \pm\frac{2\pi}{N}, \frac{3\pi}{N}, \cdots, \pm\frac{(N-1)\pi}{N}, \pm\frac{(N+1)\pi}{N}$.

Between successive principal maxima, N-1 minima occur, and between each pair of principal maxima, N-2 subsidiary maxima occur.

NEXT TIME: Two-dimensional diffraction using both rectangular and circular apertures.