LAST TIME: Review of complex algebra and its applications to physics

Complex elementary functions: sines, cosines, hyperbolic sines and cosines, exponentials, polynomials, and logarithms

For polynomials, nothing really changes except the coefficients may be complex as well. Euler's formula allows us to derive the ordinary relationships between $e^{i\theta}$ and the sines and cosines. We may then relate the hyperbolic functions to the circular functions and vice versa by using the following connections.

$$\cosh i\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta,$$

and

$$\cos i\theta = \frac{e^{i^2\theta} + e^{-i^2\theta}}{2} = \frac{e^{\theta} + e^{-\theta}}{2} = \cosh\theta\,.$$

The same substitutions for the circular trig functions and the hyperbolic trig functions yield the results given by

$$\sinh i\theta = i\sin\theta \text{ and } \sin i\theta = i\sinh\theta.$$

The logarithm function has a peculiarity. Consider $w(z) = \ln z = \ln r\, e^{i\theta} = \ln r + \ln e^{i\theta}$. Therefore, $w = \ln r + i\theta = u + iv$. The value of $e^{i\theta}$ is periodic 2π, but the value for $w(z)$ changes every time we go through 2π. The region that spans $0 \leq \theta \leq 2\pi$ is said to be the principal branch, and the x-axis forms the branch cut of the multi-valued function. More about this topic later.

Continuity and derivatives of complex variables

A real function $f(x)$ is said to be continuous at $x = a$ if, for any positive ϵ, we can choose a positive value for δ such that whenever $|h| < \delta$, then

$$|f(a+h) - f(a)| < \epsilon.$$

Roughly stated, this means that $f(x)$ is close to $f(a)$ whenever x is close to a. If we simply replace x with z, a function is continuous whenever $f(z+h)$ is close to $f(z)$. As an example, consider the function given by

$$f(z) = \frac{1}{z}.$$

To investigate the continuity of the function, let $h = \alpha + i\delta$. Then

$$f(z+h) = f(x + iy + \alpha + i\delta) = \frac{1}{x + \alpha + i(y + \delta)} = \frac{x + \alpha - i(y + \delta)}{(x+\alpha)^2 + (y + \delta)^2}.$$

Calculate $f(z+h)-f(z)$ and obtain

$$f(z+h)-f(z)=\frac{x+\alpha-i(y+\ \delta)}{(x+\alpha)^2+(y+\ \delta)^2}-\frac{x-iy}{x^2+y^2},$$

$$=\frac{(\alpha+i\delta)(x-iy)^2}{[(x+2\alpha x+y^2+2\delta y)](x^2+y^2)}=\frac{h(x-iy)^2}{[(x+2\alpha x+y^2+2\delta y)](x^2+y^2)}\to 0\ as\ h\to 0.$$

where we have used a common denominator and neglected squares of α and δ. This function is not defined at $z=0$, but it is continuous everywhere else.

Differentiability

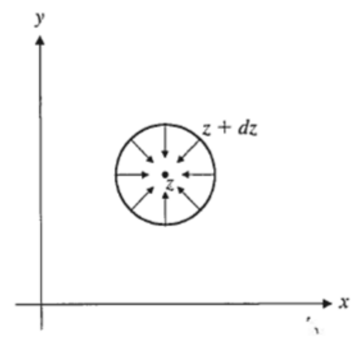
We say that a real function $f(x)$ is differentiable at $x=a$ if the derivative of the function is the same whether we approach from the LHS or the RHS. Mathematically, this means that

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(a)-f(a+h)}{h}.$$

The function $f(x)=x^2$ (a parabola that passes through the origin) is differentiable, but the function

$f(x)=x\ if\ x<0$ and $=-x\ if\ x>0$ is not.

What does this function look like? You can then see why the LHS derivative and RHS derivative are not equal. For complex variables, the definition of the derivative is the same, but why is the problem more complicated? This figure shows that instead of approaching our point from either the left or the right, we may approach it from any direction in the complex plane. Let's rewrite the definition as



$$\lim_{dz\to 0}\frac{f(z+dz)-f(z)}{dz}\bigg|_{dz=dx}=\lim_{dz\to 0}\frac{f(z+dz)-f(z)}{dz}\bigg|_{dz=i\,dy}.$$

Let $f(z)=u(x,y)+iv(x,y)$ and insert this into the equation above to get

$$\lim_{dx\to 0}\frac{u(x+dx,y)+iv(x+dx,y)-u(x,y)-iv(x,y)}{dx}$$

$$=\lim_{dy\to 0}\frac{u(x,y+dy)+iv(x,y+dy)-u(x,y)-iv(x,y)}{i\,dy}.$$

We take the indicated limits to obtain the following two equations. Note that each of the derivatives is a partial derivative in this case because of the way we defined things.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Finally, equating the real and imaginary parts yields the Cauchy-Riemann relations given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; and \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are both necessary and sufficient conditions for differentiability of complex functions.

The Cauchy-Riemann relations also have another interesting feature that we see by taking the second derivatives with respect to *x* and with respect to *y*. First, with respect to *x* and *y* gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} \; and \; \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2},$$

so we see that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The same equation holds for v as well so

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

We will see more about these Laplace equations later.

Let's look at an example to see how this process works in general.

Show that the function $f(z) = 2y + ix$ is not differentiable anywhere in the complex plane. Consider the definition of the derivative given by

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{2y + 2\Delta y + ix + i\Delta x - 2y - ix}{\Delta x + i\Delta y}$$

$$= \frac{2\Delta y + i\Delta x}{\Delta x + i\Delta y}.$$

It is not hard to see that when we take the limit as either Δx or Δy go to zero, the result is different. Is there a more general way to show that no matter which way we approach zero, we get a general result?

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta x, \Delta y \to 0} \frac{2m + i}{1 + im} \quad \text{with m as the slope of our path.}$$

Because m occurs in the real part in the numerator and in the imaginary part in the denominator, this function is nowhere differentiable in the complex plane.

Just to confirm our results, consider the results from the Cauchy-Riemann relations. Here $u(x, y) = 2y$ and $v(x, y) = x$. Therefore,

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0, \text{but } \frac{\partial v}{\partial x} = 1 \text{ and } \frac{\partial u}{\partial y} = 2.$$

Mapping

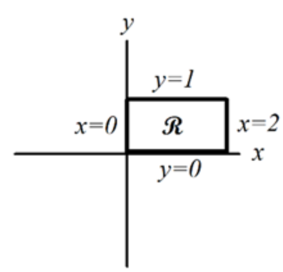
A complex function $w = f(z)$ may be thought of as mapping (or transformation) of the points (x, y) in the $x - y$ or z plane to points (u, v) in the w plane. This process is somewhat different from the real numbers because in the real number system, a function $y = f(x)$ may be thought of as mapping a value of *x* into a value of *y*. But because we need only a plane to show the two points, a graph is easy to represent the function. In the world of complex functions, however, the *z*-plane is already used to represent a single point. Whenever we think of a function such as $w = f(z) = u(x, y) + iv(x, y)$, we see that the *z*-plane will have the values of *x* and *y*, but the *w*-plane will have the values of *u* and *v*. Here are a few mappings to see what is going on here.

$$w(z) = z + \beta \Longrightarrow \text{translation}$$
$$w(z) = ze^{i\theta_o} \Longrightarrow \text{rotation}$$
$$w(z) = az \Longrightarrow \text{stretching}$$
$$w(z) = \frac{1}{z} \Longrightarrow \text{inversion}$$

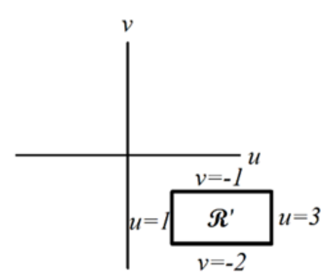
There are more general transformations we will mention in a while. Let's see how some of this works. A bit later, we will see later how some of the mappings, specifically conformal mapping, may be used to help solve problems.

Example 1

Suppose a rectangular region $\mathcal{R}$ is formed in the z-plane and is bounded by $x = 0, y = 0, x = 2,$ and $y = 1$. Determine the region $\mathcal{R}'$ in the w-plane into which $\mathcal{R}$ is mapped under the following transformations: (a) $w = z + (1 - 2i)$ and (b) $w = \sqrt{2}ze^{i\frac{\pi}{4}}$.



Solution: (a) $w = z + (1 - 2i)$ means $w = x + iy + 1 - 2i = x + 1 + i(y - 2)$ which implies that $u(x, y) = x + 1$ and $v(x, y) = y - 2$. We use the values for *x* and *y* to determine the values for *u* and *v* in the *w*-plane. Therefore $x = 0$ maps into $u = 1, y = 0$ maps into $v = -2, x = 2$ maps into $u = 3,$ and $y = 1$ maps into $v = -1.$ That means that the region $\mathcal{R}'$ in the $w -$ plane looks as follows. Notice that according to our definition of translation above, this is a pure translation as we expect because $\beta = 1 - 2i$.

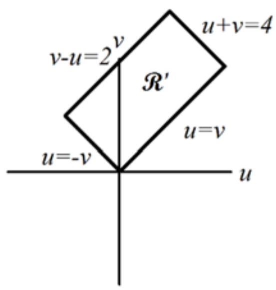


(b) $w = \sqrt{2}ze^{i\frac{\pi}{4}} = \sqrt{2}(x+iy)\,(\cos\frac{\pi}{4} + i\,\sin\frac{\pi}{4}) = \sqrt{2}(x+iy)\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = (x+iy)(1+i)$

Therefore, $u(x,y) = x - y$ and $v(x,y) = x + y$. So, $x = 0$ maps to $u = -y, v = y \;\Rightarrow u = -v$. $y = 0$ maps into $u = x, v = y, \Rightarrow u = v, x = 2$ maps into $u = 2 - y, v = 2 + y \;\Rightarrow u + v = 4$, and $y = 1$ maps into $u = x - 1, v = x + 1 \;\Rightarrow v - u = 2$.

We must construct each one of these lines in the $u - v$ plane to see what the mapping looks like.

Here is the graph for this mapping. As expected, this is a rotation by 45 degrees and a stretching by $\sqrt{2}$. Notice also that both transformations preserve angles. Such mappings are called conformal mappings. You could well say, so what! I have already shown you that *u* and *v* satisfy Laplace's equation in 2 dimensions, and that makes them harmonic functions. This means that these ideas might be applicable to fluid flow, heat flow, and electrostatics. The idea of conformal mapping can sometimes be used to transform a problem that is particularly challenging in one form to another form where it is easier to deal with. After we develop some of the integration techniques for complex functions, we will return to this problem, for now, I just want you to be familiar with the mapping process.



Analyticity

A function that is differentiable at $z = a$ and within a neighborhood of $z = a$ is analytic at $z = a$. Since a complex function represents a mapping of the z-plane onto the w-plane, any integral given by

$$\int_{z_1}^{z_2} f(z)dz$$

is an integral along a path that must be specified between the points z_1 and z_2. We may rewrite this integral by expanding the integrand and the differential to obtain

$$\int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2}(u+iv)(dx+idy) = \int_{z_1}^{z_2}[u\,dx - v\,dy + i(u\,dy + v\,dx)].$$

We define the vectors **u** and **v** by $\mathbf{u} = (\mathrm{u}, -\mathrm{v})$ and $\mathbf{v} = (\mathrm{v}, \mathrm{u})$, we may rewrite the integral as

$$\int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2}\mathbf{u}\cdot \mathrm{d}\boldsymbol{\ell} + i\int_{z_1}^{z_2}\mathbf{v}\cdot \mathrm{d}\boldsymbol{\ell}$$

We are already familiar with line integrals from our previous work. We want to consider evaluating this integral along a path from z_1 to z_2 along C_1 and back to z_1 along $C_2 -$ in other words, along a closed path. Therefore, we obtain

$$\oint f(z)dz = \oint \mathbf{u}\cdot \mathrm{d}\boldsymbol{\ell} + i\oint \mathbf{v}\cdot \mathrm{d}\boldsymbol{\ell},$$

Where all integrals are along $C = C_1 + C_2$. Sometime ago, we mentioned Green's theorem in the plane, which is given by

$$\oint (f\ dx + g\ dy) = \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy,$$

which, as I hope you remember, is just Stokes' theorem for the special case where the curve lies entirely in the *x*-*y* plane. We obtain, by assigning the proper values to f and g

$$\oint f(z)dz = \iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\ dy + i\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\ dy = 0 \text{ by the Cauchy} - \text{Riemann}$$

relations. A fundamental theorem states that if $f(z)$ is analytic in and on C

$$\oint f(z)dz = 0.$$

This is Cauchy's theorem. We will find many uses for it during our studies of complex integration.