Do examination 2 comments at end of lecture first.

Finish the last example on Green functions and give a methodology for calculating the Green function in general. Here is a fairly general method for calculating a Green function.

1. Decide what the object is, usually a sphere or cylinder, and indicate the regions where $r > r'$ and $r < r'$. It is not a bad idea to draw the configuration.

2. If you are outside a sphere, as we were in my example, the inner boundary of the region is at the surface of the sphere, and the outer boundary is at infinity.

3. Place a point charge at $\mathbf{r}'$ and write the equation $\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}')$. Put in the appropriate BCs.

4. Within the region where $\mathbf{r} \neq \mathbf{r}'$, solve the equation $\nabla^2 G(\mathbf{r}, \mathbf{r}') = 0$.

5. The region is usually divided using the function that does not have orthogonal functions. Usually, for spheres or cylinders, the dividing variable is either r or ρ.

6. Apply the BCs at the inner and outer region of interest.

7. Set the Green functions for region 1 and region 2 equal to one another at $r = r'$ or $\rho = \rho'$. This step will probably require use of the orthogonality condition for the variable has the orthogonal functions, probably spherical harmonics or Bessel functions, for sphere of cylinders, respectively.

8. By now there should be only one set of constants to determine. You should be able to write the ODE with only one variable, again either r or ρ.

9. Integrate the ODE across the boundary at $r = r'$ or $\rho = \rho'$.

Complex Analysis

Real numbers, positive and negative integers, rational numbers (numbers expressed as a ratio of two integers), and irrational numbers (numbers that cannot be expressed as the ratio of two integers) do not allow for the solution to problems of the type $x^2 + 1 = 0$. For this reason, complex numbers must be introduced to deal with such problems, and many more, for that matter.

Review complex algebra first.

A complex number is usually represented by $z = x + iy$ or $z = (x, y)$ in the Cartesian representation, where $i^2 = -1$. Its complex conjugate is given by $z^* = x - iy$, where z^* is the usual designation for the complex conjugate. The real part of the complex number is $Re\, z = x$, and the imaginary part is $Im\, z = y$. Here are the fundamental operations for complex numbers.

Addition: $(a+ib)+(c+id)=(a+c)+i(c+d)$

Subtraction: $(a+ib)-(c+id)=(a-c)+i(b-d)$

Multiplication: $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$

Division: $\frac{(a+ib)}{(c+id)}=\frac{(a+ib)(c-id)}{(c+id)(c-id)}=\frac{(ac+bd)}{(c^2+d^2)}+i\frac{(bc-ad)}{(c^2+d^2)}$

In addition, here are the rules for commutative, associative, and distributive operations.

$$z_1+z_2=z_2+z_1$$
$$z_1+(z_2+z_3)=(z_1+z_2)+z_3$$
$$z_1z_2=z_2z_1$$
$$z_1(z_2z_3)=(z_1z_2)z_3$$
$$z_1(z_2+z_3)=z_1z_2+z_1z_3$$
$$z_1+0=0+z_1$$
$$1\cdot z_1=z_1\cdot 1$$

That z_1+z_2 and z_1z_2 belong to the set is called closure.

Members of a set that satisfy the rules listed above is called a *field*, as opposed to the *group* that we mentioned earlier. Have you ever heard of a *ring*, another one of the collection of objects that have certain properties?

We may also define the absolute value, or modulus, of a complex number by the expression given by

$$|z|=(x^2+y^2)^{\frac{1}{2}}.$$

Here is a reminder of some of the rules concerning absolute values.

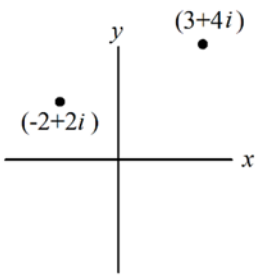
$$|z_1z_2|=|z_1||z_2|$$
$$\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$$

$|z_1+z_2|\le|z_1|+|z_2|$ and applies to any number of complex numbers

$$|z_1-z_2|\ge|z_1|-|z_2|$$

These last two inequalities are known as the triangle inequalities and state that no side of a triangle is greater in length than the sum of the other two sides nor less than the difference of the lengths of the other two sides.

Complex numbers may be easily represented graphically with the real part of the number being the *x*-component and the imaginary part of the number being the *y*-component. Here is the graph.



We may also represent the complex number in polar form by considering the following Cartesian components written as

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Therefore, $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ with $\theta = \tan^{-1}\frac{y}{x}$. Any range of angles that spans 2π may be termed the principal range, and the value of θ is called the principal value. Usually, $0 - 2\pi$ is chosen. The usual warnings for choosing the angle apply; *i.e.*, you must consider the quadrant in which the point lies. For the point $(-2 + 2i)$ shown above, $\theta = \tan^{-1}\left(\frac{2}{-2}\right)$ and $\theta = 135^o = \frac{3\pi}{4}$. To determine roots of complex numbers, we use the results from considering the product of complex numbers in polar form given by

$$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$$
$$= r_1r_2[\cos\ (\theta_1 + \theta_2) + i\sin\ (\theta_1 + \theta_2).$$

If $\theta_1 = \theta_2 = \theta\ and\ r_1 = r_2 = r$, we see that $z^2 = r^2[\cos 2\theta + i\sin 2\theta]$. By mathematical induction, we also see that

$$z^n = r^n[\cos n\theta + i\sin n\theta] = r^ne^{in\theta}.$$

This result also means that $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$, a statement known as De Moivre's theorem. If we want to extract the n^{th} root of a number, we solve the equation given by

$$z_o^n = z,$$

which implies that we need to solve

$$r_o^n[\cos n\theta_o + i\sin n\theta_o] = r(\cos\theta + i\sin\theta).$$

This means that

$$r_o^n = r\ and\ n\theta_o = \theta + 2k\pi$$

and

r_o is the n^{th} root of r and $\theta_o = \frac{\theta}{n} \pm \frac{2k\pi}{n}$. Our expression for z_o becomes

$$z_o = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right).$$

As an example, suppose we wish to find the fifth root of -32. We write

$-32 = 32[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$ to put the number in the correct complex form. Then, according to De Moivre's theorem,

$$z^n = r^n[\cos n\theta + i\sin n\theta] = 32[\cos 5\theta + i\sin 5\theta]$$

from which

$$z = 2\left[\cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)\right].$$

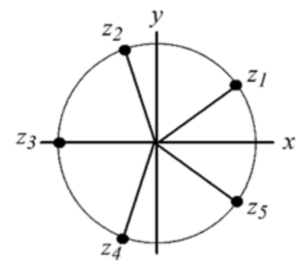
For $k = 0,1,2,3,4$, we generate the 5 fifth roots of -32 to be

$$z_1 = 2\left[\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right], \quad z_2 = 2\left[\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right],$$

$$z_3 = 2\left[\cos\left(\frac{5\pi}{5}\right) + i\sin\left(\frac{5\pi}{5}\right)\right] = -2, \quad z_4 = 2\left[\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right],$$

and

$$z_5 = 2\left[\cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)\right].$$



On the complex plane, these five roots are represented by the points on an equilateral pentagon as shown to the right. Notice that the graphical representation shows right away the real root at -2, and the other two sets of complex conjugate roots, z_1 and z_5, and z_2 and z_4. Solutions to the quadratic equation using complex coefficients is no different than with real coefficients. Here is a short example. Determine the values of z for the equation $z^2 + (2i - 3)z + 5 - i = 0$. Here $a = 1, b = (2i - 3), and\ c = 5 - i$. Using the quadratic formula gives

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(2i - 3) \pm \sqrt{(2i - 3)^2 - 4(1)(5 - i)}}{2}$$

$$= \frac{-2i + 3 \pm \sqrt{-4 - 12i + 9 - 20 + 4i}}{2} = \frac{-2i + 3 \pm \sqrt{-15 - 8i}}{2}.$$

Now you see why we need to be sure we know how to take roots of complex numbers because of what occurs under the square root here. Let's look at a second method that is particularly useful for square roots. Let $a + ib$ represent the resulting square root so that $a + ib = \sqrt{-15 - 8i}$. We may square each side to obtain

$$a^2 - b^2 + 2iab = -15 - 8i.$$

Equating the real parts and imaginary parts gives $a^2 - b^2 = -15$ and $2ab = -8$ so $ab = -4$.

Setting $b = -\frac{4}{a}$ and substituting it into the second equation gives the quadratic equation in a^2 to be $a^4 + 15a^2 - 16 = 0$, which implies that $(a^2 + 16)(a^2 - 1) = 0$. Because a is real, $a = \pm 1$. If $a = +1, b = -4\ but\ if\ a = -1, b = +4$. Therefore, the correct solutions are given by

$$z = \frac{-2i + 3 \pm (1 - 4i)}{2} = 2 - 3i, 1 + i.$$

You probably remember using this technique when you dealt with wave propagation in a medium where the magnitude of the propagation number *k* has both real and imaginary parts. The real part represents the absorption of the wave, and the imaginary part represents the relative phase shift of the wave. Here is how it works briefly. Assume you are considering an electromagnetic wave whose amplitude is given by

$$E_x = E_{ox}e^{i(kz-\omega t)}.$$

How would you characterize this wave? Suppose we found by solving the wave equation with absorption incorporated into the equation that $k = k_R + ik_I$, where k_R and k_I are the real and imaginary parts of the complex k. Inserting the complex value back into the expression for E_x yields

$$E_x = E_{ox}e^{i[(k_R+ik_I)z-\omega t)} = E_{ox}e^{-k_I z}e^{i(k_R z-\omega t)}.$$

From the form of this equation, you see that the imaginary part of k indicates that the wave energy is being absorbed, whereas the real part gives the propagation number. The exact form of k_R and k_I must, of course, be determined by the dispersion relation being considered.

Another nice application of the complex representation of waves occurs in the study of states of polarization of EM waves. If you have studied polarization effects in some detail, you are aware that materials may cause different effects on different components of a wave. Usually, such effect are described mathematically by considering two components of a wave, the *x*- and *y*- components if the wave travels in the *z*-direction. The total propagating electric field may be written as

$$\mathbf{E} = E_{ox}e^{i(kz-\omega t)}\hat{\mathbf{x}} + E_{oy}e^{i(kz-\omega t+\theta)}\hat{\mathbf{y}}.$$

Notice that these fields have the common feature for the propagating part, but the *y*-component is shifted in phase relative to the *x*-component. This shift in phase typically occurs because the wave is propagating in an anisotropic medium where the *y*-component travels faster or slower than the *x*-component. The exact state of polarization is determined by the relative phase and by the relative magnitudes of E_{ox} and E_{oy}. Certain features might be completely clear to you, but others not so clear without further investigation. For example, if $\theta = 0$ or a multiple of 2π, the wave is linearly polarized, and the angle of the field is given by the usual vector considerations. If $E_{ox} = E_{oy}$, and the phase is $\pm\frac{\pi}{2}$, we have circular polarization. If E_{ox}, E_{oy}, and θ are all arbitrary, but not zero, we have elliptically polarized light with the major axis of the ellipse not generally aligned with either the *x* - or *y* - axis. If the waves under consideration are all monochromatic, and therefore, coherent, it is possible to give a very succinct representation of these effects using a matrix formulation. The two components of the waves are represented as column vectors given by

$$\begin{bmatrix} E_{ox} \\ E_{oy}e^{i\theta} \end{bmatrix}.$$

The effects of some of the normally used optical elements, polarizers and retarders, are given by the appropriate 2 × 2 matrices, Jones matrices, that operate on the column vectors to predict the new state of polarization after the wave transmits through the element.

One more important use of complex variables occurs in the solution of differential equations when you make the assumption that a solution may be written as $x(t) = e^{qt}$. Substituting this result back into the ODE such as the one for a damped harmonic oscillator again yields a solution that contains both an amplitude and a phase factor. When the oscillator is driven by a harmonic source, the phase lets one see how the driving force and the response to it behave

Midterm examination 2 comments

A total of 6 problems will be on the examination. Expect 4 or 5 boundary value problems covering Cartesian coordinates, cylindrical coordinates with no ends, cylindrical coordinates with ends, spherical coordinates with azimuthal symmetry, spherical coordinates without azimuthal symmetry, and 1 or 2 Green function problems. There may not be a problem on each of these coordinates, but every problem will be chosen from these systems. I will put on the exam all orthogonality conditions and give you the behavior of any special functions you might need to know. You may also bring one 8.5” x 11” sheet of paper with writing on both sides.