LAST TIME: Bessel functions and Green functions

Recall the basic idea of a Green function is the following. Our input is represented as

$$I(x) = \int_0^L I(x')\delta(x - x')dx',$$
$$y(x) = \int_0^L I(x')G(x, x')dx',$$

where we solve the same differential equation with a unit impulse as the source.

As an example, let's use the Green function for a Sturm-Liouville ODE that we found at the end Lecture 14 to solve a problem. The one-dimensional Helmholtz equation is a Sturm-Liouville ODE given by

$$\frac{d^2y}{dx^2} + k^2 y = f(x),$$

where $f(x)$ is to be specified at a later time, but is known over a range $0 < x < L$. The Green function ODE is given by

$$\frac{d^2G}{dx^2} + k^2 G = \delta(x - x').$$

Because this equation is a Sturm-Liouville ODE, we only need to calculate the eigenfunctions for $y(x)$. Consider

$$\frac{d^2y_n}{dx^2} + k^2 y_n = 0$$

to find the eigenfunctions

$$y_n(x) = C_n \sin\sqrt{\lambda_n}\, x$$

with

$$\sqrt{\lambda_n}L = n\pi.$$

I have assume homogeneous BCs as required for Sturm-Liouville systems. The eigenvalues are given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

The normalization constant is obtained in the standard way and is given by

$$C_n = \sqrt{\frac{2}{L}}.$$

The normalized eigenfunctions are given by

$$y_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}.$$

The Green function is constructed to be

$$G(x,x') = \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x'}{L}\right) \sin\left(\frac{n\pi x}{L}\right)}{(kL/\pi)^2 - n^2}.$$

Notice that the expression cannot apply when the denominator goes to zero. What does this correspond to physically? We may insert a value for $f(x)$ to get the solution for $y(x)$. Let

$$f(x) = 1 \; for \frac{L}{4} < x < \frac{3L}{4}.$$

This gives a value for $y(x)$ of

$$y(x) = \frac{2}{L} \int_{L/4}^{3L/4} \sum_n \frac{\sin\left(\frac{n\pi x'}{L}\right) \sin\left(\frac{n\pi x}{L}\right)}{(k)^2 - \left(\frac{n\pi}{L}\right)^2} dx'$$

$$= \frac{2}{L} \sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)}{(k)^2 - \left(\frac{n\pi}{L}\right)^2} \int_{L/4}^{3L/4} \sin\left(\frac{n\pi x'}{L}\right) dx'$$

$$= \frac{2}{L} \sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)}{(k)^2 - \left(\frac{n\pi}{L}\right)^2} L \frac{\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}}{n\pi}$$

$$= 2 \frac{\sin\left(\frac{n\pi x}{L}\right)}{-(k)^2 + \left(\frac{n\pi}{L}\right)^2} \frac{2 \sin n\pi/2 \sin n\pi/4}{n\pi}$$

$$= 4 \sum_{n\; odd}^{\infty} (-1)^{\frac{n-1}{2}} \left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{-(k)^2 + \left(\frac{n\pi}{L}\right)^2}\right] \frac{\sin n\pi/4}{n\pi}.$$

To understand the significance of this result, recall the origin of the Helmholtz. It comes from assuming a harmonic time dependent solution to the wave equation. This result is the result of an initial condition that sets up a standing wave of the form indicated.

Let's consider how to solve problems involving Poisson's equation given by

$$\nabla^2 V = -\frac{\rho(\mathbf{r})}{\epsilon_o}.$$

The usual way to develop the Green function in this case is to consider two scalar functions V and Ψ defined in a volume $d^3\mathbf{r}$. Consider the divergence of the function $V\boldsymbol{\nabla}\Psi$ given by

$$\boldsymbol{\nabla}\cdot V\boldsymbol{\nabla}\Psi = \boldsymbol{\nabla}V\cdot\boldsymbol{\nabla}\Psi + \nabla^2\Psi.$$

We may interchange V and Ψ to obtain

$$\boldsymbol{\nabla}\cdot\Psi\boldsymbol{\nabla}\mathrm{V} = \boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}\mathrm{V} + \nabla^2\mathrm{V}.$$

Subtract the two equations, integrate over the volume, and use the divergence theorem to obtain

$$\int (V(\mathbf{r}')\nabla^2\Psi - \Psi\nabla^2\mathrm{V}(\mathbf{r}'))d^3\mathbf{r}' = \oint (V(\mathbf{r}')\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\mathrm{V}(\mathbf{r}'))\cdot\mathbf{n}\,dS'.$$

This equation is known as Green's theorem or Green's second identity. It is valid for any scalar functions with derivatives that exist in the volume and on the surfaces. We proceed to make the appropriate substitutions to obtain

$$\int\left(-4\pi\delta(\mathbf{r}-\mathbf{r}')V(\mathbf{r}') + G(\mathbf{r},\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon_o}\right)d^3\mathbf{r}' = \oint (V(\mathbf{r}')\boldsymbol{\nabla}G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\boldsymbol{\nabla}\mathrm{V}(\mathbf{r}'))\cdot\mathbf{n}\,dS',$$

where I have used $\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$ with $\Psi(\mathbf{r},\mathbf{r}') = G(\mathbf{r},\mathbf{r}')$. Sometimes, you will see $\boldsymbol{\nabla}G(\mathbf{r},\mathbf{r}')\cdot\mathbf{n} = \frac{\partial G}{\partial n}$ referred to as the normal derivative. If the boundary condition specifies V on the surface, we choose $G_D(\mathbf{r},\mathbf{r}') = 0$ on the surface to obtain

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\int G_D(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')d^3\mathbf{r}' - \frac{1}{4\pi}\oint V(\mathbf{r}')\,\boldsymbol{\nabla}G_D(\mathbf{r},\mathbf{r}')\cdot\mathbf{n}dS'.$$

This equation tells us how to calculate the potential arising from a charge distribution with certain boundary conditions, but we need to know the Green function to make it useful. We find the Green function for the region outside a sphere of radius R. This is done in the usual manner where we solve the equation (point charge at $\mathbf{r}'$)

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}'),$$

where the BCs are $G(\mathbf{r},\mathbf{r}') = 0$ for $r = R$ and $r\to\infty$. We divide the outside region into two sections, one where $r > r'$ and one where $r < r'$. As long as we are not at the charge, the equation to solve is given by

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = 0,$$

and the solution is given by

$$G(\mathbf{r},\mathbf{r}') = \sum_{\ell,m} \left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\phi).$$

The two regions are defined by

$$Region\ I: R < r < r'$$

and

$$Region\ II: r' < r < \infty.$$

For region II, *G* must go to zero so its solution is given by

$$G_{II}(\mathbf{r},\mathbf{r}') = \sum_{\ell,m} \left(\frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\phi).$$

At $r = R$, we must have

$$A_{\ell m} R^{\ell} + \frac{B_{\ell m}}{R^{\ell+1}} = 0,$$

so

$$B_{\ell m} = -R^{2\ell+1} A_{\ell m}.$$

Therefore,

$$G_I(\mathbf{r},\mathbf{r}') = \sum_{\ell,m} \left(A_{\ell m} r^{\ell} - \frac{R^{2\ell+1}}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\phi).$$

At $r = r'$, we match the two Green functions to obtain

$$\sum_{\ell,m} \left(A_{\ell m} (r')^{\ell} - \frac{R^{2\ell+1}}{(r')^{\ell+1}} \right) Y_{\ell m}(\theta,\phi) = \sum_{\ell,m} \left(\frac{B_{\ell m}}{(r')^{\ell+1}} \right) Y_{\ell m}(\theta,\phi).$$

We exploit the orthogonality condition for the spherical harmonics by multiplying each side of the equation by $Y^*_{\ell' m'}(\theta,\phi)$. This will show that each coefficient must be separately equal, and, as a result

$$B_{\ell m} = A_{\ell m}\left[(r')^{\ell+1} - R^{2\ell+1}\right].$$

We may write the Green function in terms of a single constant which will have to be determined by integrating the differential equation, just as we did in the earlier problem. We use this result to write an expression for $G_{II}(\mathbf{r},\mathbf{r}')$ given by

$$G_{II}(\mathbf{r},\mathbf{r}') = \sum_{\ell,m} \left(\frac{A_{\ell m}\left[(r')^{2\ell+1} - R^{2\ell+1}\right]}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\phi).$$

Set $A_{\ell m} = \frac{\alpha_{\ell m}}{(r')^{\ell+1}}$ and substitute into $G_I(\mathbf{r},\mathbf{r}')$ and $G_{II}(\mathbf{r},\mathbf{r}')$ to obtain

$$G_I(\mathbf{r},\mathbf{r}') = \sum_{\ell,m} \frac{\alpha_{\ell m}}{(r')^{\ell+1} r^{\ell+1}} \left(r^{2\ell+1} - R^{2\ell+1}\right) Y_{\ell m}(\theta,\phi)$$

and

$$G_{II}(\mathbf{r},\mathbf{r}') = \sum_{\ell,m} \frac{\alpha_{\ell m}\left[(r')^{2\ell+1} - R^{2\ell+1}\right]}{(r')^{\ell+1} r^{\ell+1}} Y_{\ell m}(\theta,\phi).$$

We have to obtain the value for the unknown constant still lingering. To do this, we have to return to the original differential equation and integrate across the boundary at $r = r'$. This will involve matching the two values for $G(\mathbf{r},\mathbf{r}')$ at $r = r'$. I will only outline the process without dragging you through the gory details. (I am using partly what is in Susan Lea's book for this development.) We make the substitution $G(\mathbf{r},\mathbf{r}') = \sum g_{\ell m} Y_{\ell m}$ and express the Dirac delta function in spherical coordinates as follows: $\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(r-r')}{r^2}\delta(\mu-\mu')\delta(\phi-\phi')$, where the r^{-2} is necessary to keep the integral over the volume correct. This process yields

$$\sum_{\ell,m} \frac{1}{r}\frac{d^2(r g_{\ell m})}{dr^2} Y_{\ell m} - \frac{\ell(\ell+1)}{r^2} Y_{\ell m} g_{\ell m} = -4\pi \frac{\delta(r-r')}{r^2}\delta(\mu-\mu')\delta(\phi-\phi').$$

We eliminate the $Y_{\ell m}$'s by multiplying by $Y^*_{\ell' m'}$ and using the orthogonality condition to obtain

$$\frac{1}{r}\frac{d^2(r g_{\ell m})}{dr^2} - \frac{\ell(\ell+1)}{r^2} g_{\ell m} = -4\pi \frac{\delta(r-r')}{r^2} Y^*_{\ell' m'}.$$

Now we multiply by r and integrate across the boundary at $r = r'$. We need to remember that on one side of the boundary, we will use $G_{II}(\mathbf{r},\mathbf{r}')$ and $G_I(\mathbf{r},\mathbf{r}')$ on the other side. Here is the integral.

$$\int_{r'-\epsilon}^{r'+\epsilon} \frac{d^2(r f_{\ell m})}{dr^2} dr - \frac{\ell(\ell+1)}{r} f_{\ell m} dr = -4\pi \int_{r'-\epsilon}^{r'+\epsilon} \frac{\delta(r-r')}{r} dr.$$

I have substituted $g_{\ell m} = f_{\ell m} Y^*_{\ell' m'}$ and $\alpha_{\ell m} = \beta_{\ell m} Y^*_{\ell m}$ to make things come out with only $\beta_{\ell m}$ as unknown. The second term on the LHS is zero in the limit of ϵ going to zero. The RHS becomes $-4\pi/r'$. We are then left with

$$\frac{d}{dr}(r f_{\ell m})\Big|_{r'_-}^{r'_+} = -\frac{4\pi}{r'}.$$

We now evaluate these derivatives and equate the two values for $G(\mathbf{r},\mathbf{r}')$ at the boundary. Wading through the derivatives (see Lea's book ir you wish to see more details) gives a value for $\beta_{\ell m}$ of

$$\beta_{\ell m} = \frac{4\pi}{2\ell+1}.$$

The usual way or writing everything in one formula is to use the terminology $r_<$ and $r_>$ to stand for the lesser and greater values of r and r'. Doing so finally gives a value for the Green function of

$$G(\mathbf{r}, \mathbf{r}') = \sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^{2\ell+1} - R^{2\ell+1}}{r_<^{\ell+1} r_>^{\ell+1}} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi').$$

Having waded through this algebra, what can we do with this expression now? What we have now is an expression that will allow us to calculate the potential due to a grounded sphere with an arbitrary charge distribution located outside the sphere. We can calculate the potential both inside and outside where the charge is located by assigning the correct terms to the values in the equation above. Here is an example of how to use this expression.

Suppose a ring of charge having radius $b > R$ and constant linear charge density λ lies in the equatorial plane of a grounded sphere having radius R. We wish to know the potential everywhere outside the sphere. This entire Green function goes into the expression for $V(r)$ that we derived earlier given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_o} \int G_D(\mathbf{r}, \mathbf{r}')\, \rho(\mathbf{r}') d^3\mathbf{r}' - \frac{1}{4\pi} \oint V(\mathbf{r}')\, \boldsymbol{\nabla} G_D(\mathbf{r}, \mathbf{r}') \cdot \mathbf{n} dS'.$$

Because we have declared the sphere to be grounded, the second integral is zero, leaving only the integral over $G(\mathbf{r}, \mathbf{r}')$ and $\rho(\mathbf{r}')$. We must express $\rho(\mathbf{r}')$ in the correct way so that its integral over its volumes leaves us with the total charge. The total charge Q must come out to be $2\pi b\lambda$. The Dirac delta function representation of the charge density is given by

$$\rho(\mathbf{r}') d^3\mathbf{r}' = \lambda \frac{\delta(r'-b)}{b} \delta(\mu') r^{2'} dr' d\mu' d\phi' \text{ with } \rho(\mathbf{r}') = \lambda \frac{\delta(r'-b)}{b} \delta(\mu').$$

You can see that the integral gives the correct total charge of $2\pi b\lambda$. Therefore,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_o} \int \lambda \frac{\delta(r'-b)}{b} \delta(\mu') \times$$

$$\sum_{\ell,m} \frac{4\pi}{2\ell+1} \frac{r_<^{2\ell+1} - R^{2\ell+1}}{r_<^{\ell+1} r_>^{\ell+1}} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi') r^{2'} dr' d\mu' d\phi'$$

$$= \frac{\lambda}{\epsilon_o} \sum_{\ell,m} \frac{1}{2\ell+1} Y_{\ell m}(\theta, \phi) b \frac{r_<^{2\ell+1} - R^{2\ell+1}}{r_<^{\ell+1} b^{\ell+1}} \frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(0) \int_0^{2\pi} e^{-im\phi'}\, d\phi'.$$

The integral over ϕ' is zero unless $m = 0$. This gives an enormous simplification so that

$$V(\mathbf{r}) = \frac{\lambda b}{2\epsilon_o} \sum_\ell \frac{r_<^{2\ell+1} - R^{2\ell+1}}{r^{\ell+1} b^{\ell+1}} P_\ell(0) P_\ell(\mu).$$

NEXT TIME: More examples