LAST TIME: More on Jones vectors and matrices, began interference

We begin by adding two waves having the same frequency with a relative phase of ϵ. The two waves are given by

$$\mathbf{E}_1 = \mathbf{E}_{o1} e^{i(kr_1 - \omega t)}$$

and

$$\mathbf{E}_2 = \mathbf{E}_{o2} e^{i(kr_2 - \omega t + \epsilon)}.$$

Therefore, the total field is given by

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_{o1} e^{i(kr_1 - \omega t)} + \mathbf{E}_{o2} e^{i(kr_2 - \omega t + \epsilon)}.$$

One very big advantage in working with the exponential form of the electric field is that it is not hard to prove that the irradiance I is given by

$$I = \frac{1}{2} Re(\mathbf{E} \cdot \mathbf{E}^*), \text{which comes from } \langle \text{fg} \rangle = \frac{1}{2} Re(f g^*)$$

Therefore,

$$I = \frac{1}{2} Re\{\mathbf{E}_{o1} e^{i(kr_1 - \omega t)} \cdot \mathbf{E}_{o1} e^{-i(kr_1 - \omega t)} + \mathbf{E}_{o1} e^{i(kr_1 - \omega t)} \cdot \mathbf{E}_{o2} e^{-i(kr_2 - \omega t + \epsilon)} + \\ + \mathbf{E}_{o1} e^{-i(kr_1 - \omega t)} \cdot \mathbf{E}_{o2} e^{i(kr_2 - \omega t + \epsilon)} + \mathbf{E}_{o2} e^{i(kr_2 - \omega t + \epsilon)} \cdot \mathbf{E}_{o2} e^{i(kr_2 - \omega t + \epsilon)}\}$$

The first thing to notice is that the fields must have some parallel components or the cross terms will vanish leaving us with only

$$I = \frac{1}{2} Re(E_{o1}^2 + E_{o2}^2),$$

which is just $I_1 + I_2$. It is customary to assume that the two fields are parallel so we obtain

$$I = \frac{1}{2} Re\left(E_{o1}^2 + E_{o2}^2 + E_{o1} E_{o2} e^{ik(r_1 - r_2 + \epsilon)} + E_{o1} E_{o2} e^{-ik(r_1 - r_2 + \epsilon)}\right).$$

Finally,

$$I = \frac{1}{2} Re\{E_{o1}^2 + E_{o2}^2 + 2E_{o1} E_{o2} \cos[k(r_1 - r_2) + \epsilon]\}$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos[k(r_1 - r_2) + \epsilon].$$

The total phase difference breaks up neatly into two components, the phase difference due to the optical path difference $[k(r_1 - r_2)]$ and the remaining phase ϵ, which comes from the inherent phase difference + the phase differences due to reflections. I believe this is the most effective way to consider phase in interference problems. Usually, in optics, an inherent phase difference is difficult to build in, but in electronically controlled devices, it is relatively easy to do. We call the total phase difference $\delta = \delta_{OPD} + \delta_{in} + \delta_{ref}$. We note the following:

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}; \;\; \delta = 0, \pm 2\pi, \cdots \pm 2n\pi$$

and

$$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}; \;\; \delta = \pm\pi, \cdots \pm (2n+1)\pi$$

The fringe visibility is defined to be

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2},$$

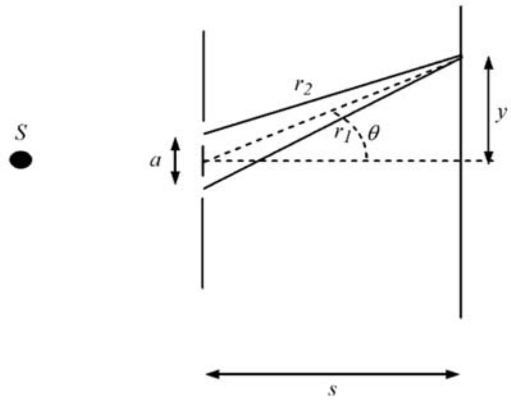
so

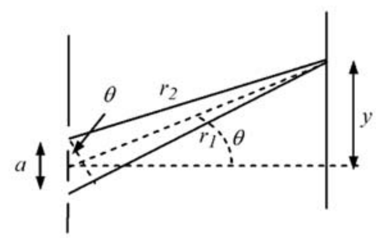
$$V = 1$$

if

$$I_1 = I_2 = I_o.$$

Young's double pinhole or double slit experiment is the most common interference for two-wave, wavefront division. The figure shows the normal arrangement. For optical interference, $s >> a$ with $s \approx 50$ cm and $a \approx 0.5$ mm. The source S is far enough away that plane waves are assumed to be incident on the slits. The second figure shows a bit more detail, so we can see that $r_1 - r_2 \cong a \sin\theta$. This is like having an isosceles triange with two very long legs so that the ange between each leg and the base is very close to 90 degrees. Here, $\epsilon = 0$, so the only phase difference results from a difference in the OPLs of the two legs. Usually, $\sin\theta \cong \tan\theta = \frac{y}{s}$, and $I_1 = I_2 = I_o$ so $I(\theta) = 2I_o + 2I_o\cos\delta = 4I_o\cos^2(\delta/2)$. But





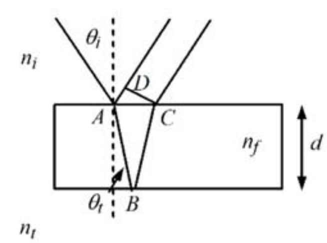
$$\delta = kr_1 - r_2 = \frac{2\pi}{\lambda} OPD,$$

so $I(y) = 4I_o \cos\frac{2\pi}{\lambda}\frac{ay}{2s} = 4I_o\cos^2\left(\frac{ya\pi}{\lambda s}\right)$.

The maxima occur whenever the argument of the cosine function is given by $m\pi$. Therefore,

$\frac{y_m a\pi}{\lambda s} = m\pi$ which implies that $y_m = \frac{m\lambda s}{a}$. The constructive interference fringes occur at equally spaced intervals given by $y_{m+1} - y_m = \frac{(m+1)\lambda s}{a} - \frac{m\lambda s}{a} = \frac{\lambda s}{a}$.

The most common example of interference with amplitude division is reflection from a plate, usually a thin film. The figure shows how this comes about. The two waves that are interfering are the one reflected from the top surface and the one that is reflected from the bottom surface and transmits back out the top surface. We may use the Fresnel equations to argue that these two waves are very close to the same amplitude. I will go through the argument in class, but see if you can determine how it works before class.



The optical path difference between those two waves is given by

$$OPD = n_f \overline{AB} + n_f \overline{BC} - n_i \overline{AD}.$$

Note that

$$\overline{AB} = \overline{BC} = \frac{d}{\cos\theta_t}; \quad \overline{AD} = \overline{AC}\sin\theta_i; \quad \frac{\overline{AC}}{2} = \tan\theta_t.$$

Therefore,

$$\overline{AD} = 2d\tan\theta_t \frac{n_f}{n_i}\sin\theta_t,$$

so

$$OPD = \frac{2n_f d}{\cos\theta_t} - \frac{n_f}{n_i}\frac{2d\sin^2\theta_t}{\cos\theta_t} n_i$$

and

$$OPD = \frac{2n_f d}{\cos\theta_t}\cos^2\theta_t = 2n_f d\cos\theta_t.$$

The total phase difference is given by

$$\delta = \frac{2\pi}{\lambda}\left(2n_f d\cos\theta_t\right) + \epsilon = \frac{4\pi n_f d\cos\theta_t}{\lambda} + \epsilon$$

Here, the phase differences by inherent differences and reflections must be considered carefully. Both waves undergo one reflection, but a phase different might occur. To determine the phase changes due to reflections, we must consider the values for n_i, n_f, and n_t. Consider first the case when the film (plate) is in air. The reflection off the front surface is an external reflection (small to larger refractive index) and does involve a phase change of π. The reflection from the back surface, however, is an internal reflection (high to low refractive index) and does not involve a phase change. Therefore, for this case,

$$\delta = \frac{4\pi n_f d\cos\theta_t}{\lambda} \pm \pi.$$

If, however, $n_t > n_f > n_i$, then both reflections would be external and would either cancel or give $2\,\pi$, which does not really matter. Any time both reflections are either internal or external, we would not need to consider the phase change from reflection. However, if one is external and one is internal, there will be a π phase change that must be used the in total phase change equation. Our condition for maxima is

$$\delta = \frac{4\pi n_f d\cos\theta_t}{\lambda} \pm \pi = 2m\pi$$

and

$$d\cos\theta_t = (2m+1)\frac{\lambda}{4n_f} = \frac{(2m+1)}{4}\lambda_f.$$

Minima will occur at

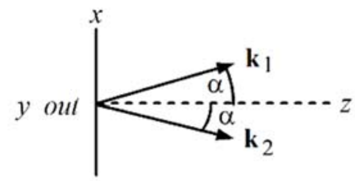
$$\delta = \frac{4\pi n_f d\cos\theta_t}{\lambda} \pm \pi = (2m+1)\pi$$

and

$$d\cos\theta_t = 2m\frac{\lambda_f}{4}.$$

For both internal or both external reflections, the situations are reversed.

Now, let's consider interference between two plane waves whose propagation vectors make an angle of 2α with respect to one another. We must first write the form of the waves that are given by



$$\mathbf{E}_1 = \boldsymbol{E}_{o1}e^{i(\mathbf{k}_1\cdot\mathbf{r}-\omega t)} \text{ and } \mathbf{E}_2 = \mathbf{E}_{o2}e^{i(\mathbf{k}_2\cdot\mathbf{r}-\omega t)}.$$

Now, write the expressions for the k's as

$$\mathbf{k}_1 = k_1\sin\alpha\,\hat{\mathbf{x}} + k_1\cos\alpha\,\hat{\mathbf{z}} \text{ and } \mathbf{k}_2 = -k_2\sin\alpha\,\hat{\mathbf{x}} + k_2\cos\alpha\,\hat{\mathbf{z}}.$$

Now, calculate I to obtain

$$I = \frac{1}{2}Re\,(\mathbf{E}_1+\mathbf{E}_2)\cdot(\mathbf{E}_1^*+\mathbf{E}_2^*)$$

$$= \frac{1}{2}Re\left\{\begin{matrix}\left[\boldsymbol{E}_{o1}e^{i(k_1\,\mathrm{x}\sin\alpha + k_1 z\cos\alpha)} + \mathbf{E}_{o2}e^{i(-k_2\,\mathrm{x}\sin\alpha + k_2 z\cos\alpha)}\right]\cdot \\ \left[\boldsymbol{E}_{o1}e^{-i(k_1\,\mathrm{x}\sin\alpha + k_1 z\cos\alpha)} + \mathbf{E}_{o2}e^{-i(-k_2\,\mathrm{x}\sin\alpha + k_2 z\cos\alpha)}\right]\end{matrix}\right\}.$$

Therefore, using $k_1 = k_2 = k$, we obtain

$$I = \frac{1}{2}\left[E_{o1}^2 + E_{o2}^2 + \boldsymbol{E}_{o1}\cdot\mathbf{E}_{o2}e^{2ik\,\mathrm{x}\sin\alpha} + \boldsymbol{E}_{o2}\cdot\mathbf{E}_{o1}e^{-2ik\,\mathrm{x}\sin\alpha}\right]$$
$$= I_1 + I_2 + 2\boldsymbol{E}_{o1}\cdot\mathbf{E}_{o2}\cos(2k\,\mathrm{x}\sin\alpha).$$

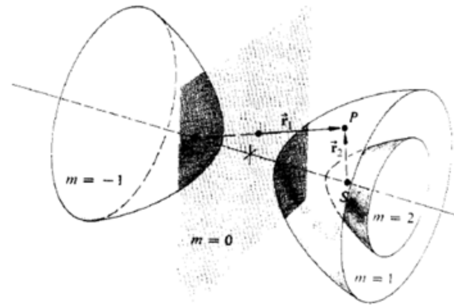
The total phase difference is given by $\delta = 2k\,\mathrm{x}\sin\alpha$, so maxima occur at $\delta = 2m\pi$ and minima occur at $\delta = (2m+1)\pi$. We convert to wavelength and write

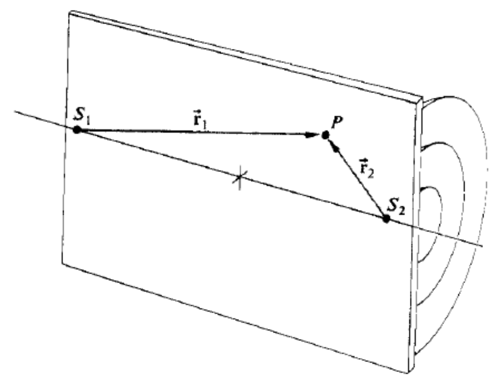
$$\frac{2(2\pi)}{\lambda}\sin\alpha\;x_m = 2m\pi \Longrightarrow\; x_m = \frac{m\lambda}{2\,\sin\alpha}.$$

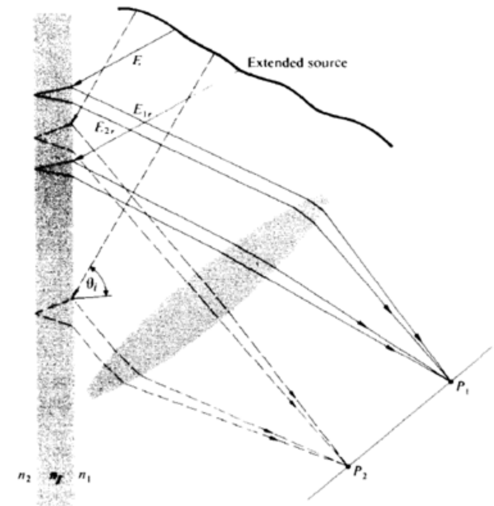
The spacing between fringes is then given by $\Delta x = x_{m+1} - x_m = \frac{\lambda}{2\,\sin\alpha}$. Comment on this result.

One of the most instructive figures to visualize two-wave interference is shown on the next page. Depending on where you observe the superposition, it contains much of the relevant information concerning two-wave interference.

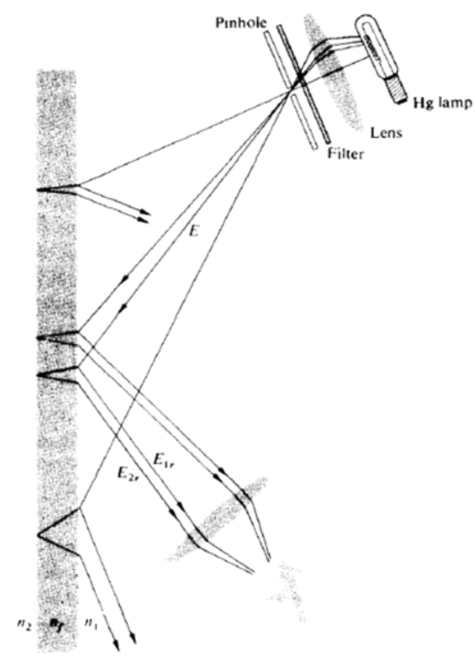
Two point sources lie along an arbitrary axis as shown. When the difference between r_1 and r_2 is equal to an interger number of wavelengths, we get constructive interference. Therefore, $r_1 - r_2 = m\lambda$. In case you have forgotten, this is the definition of a hyperbola, and in three dimensions, we get hyperloids of revolutions. If you look along the axis formed by S_1 and S_2, you will see a set of concentric circles representing constructive interference. If, on the other hand, you look along a line perpendicular to the line joining the two sources, you will see a set of almost straight lines that look like the fringes from Young's interference.

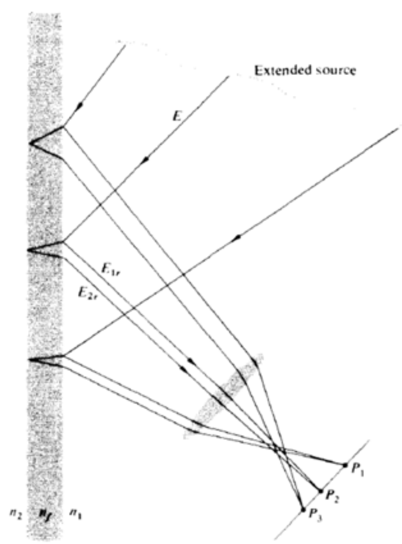






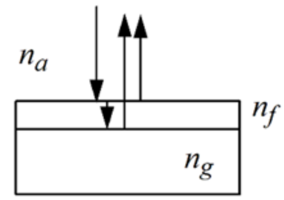
Exactly how an interference pattern appears depends on the source and how the interference is observed. Here are some figures from your text showing some of these features.





One very important application of interference is an anti-reflection coating. Most decent quality optical systems today will include this feature.

Here is how it works. A thin film is placed on top of a glass substrate. We want to adjust the film's thickness and refractive so that reflections from the air-film interface and the film-substrate come as close as possible to canceling one another in the reflected light. Usually, $n_g > n_f > n_a$. This means that each reflection is an external reflection with a phase change of π. Therefore, their effects cancel and we need only consider the phase difference from the OPD. We will adjust the wavelength for the middle of the visible spectrum – about 550 nm. To get destructive interference for the reflected waves, we want $\delta = (2m+1)\pi$. The optical path difference is 2kd, where d is the thickness of the film, so this means that $(2m+1)\pi = 2\frac{2\pi}{\lambda}n_f d$. Therefore, $d = \frac{(2m+1)\lambda}{4n_f}$. For the smallest thickness, $m = 0$, so $d = \frac{\lambda}{4n_f} = \frac{\lambda_f}{4}$. This takes care of the phase considerations, but how about the amplitudes of the waves? We want the amplitudes of the waves to be as close as possible, so we recall the Fresnel coefficients and equate the amplitude of the first reflection to that of the second reflection, thereby giving



$$\frac{n_f - n_a}{n_f + n_a} = \frac{n_g - n_f}{n_g + n_f} \Longrightarrow (n_f - n_a)(n_g + n_f) = (n_f + n_a)(n_g - n_f).$$

Cancellation of several of the terms yields the expression $n_f = \sqrt{n_a n_g}$, which is 1.22. It is hard to find a material exactly like that, so usually MgF_2, which has a refractive index of 1.38, is used.

How to tell if something is antireflection coated.

NEXT TIME: Interferometers and their uses and start multiple-wave interference.