LAST TIME: Bessel functions and Green functions

Although we have not yet developed the contour integral method, let me go ahead and set up the transform method for finding the Green function for a damped, driven harmonic oscillator.

Recall the equation for the damped and driven harmonic oscillator is given by

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F(t).$$

To put the equation in a more standard form, let's divide through by m and define some new constants which should be familiar to you.

$$\frac{d^2x(t)}{dt^2} + 2\gamma\frac{dx(t)}{dt} + \omega_o^2 x(t) = f(t).$$

We will have some initial conditions on this equation, and we will worry about those later. The Green function we need to find satisfies the equation given by

$$\frac{d^2G(t,t')}{dt^2} + 2\gamma\frac{dG(t,t')}{dt} + \omega_o^2 G(t,t') = \delta(t-t').$$

Equations of this type may be solved using a Fourier transform to convert the ODE to an algebraic equation, much in the same way you used the substitution of e^{qt} to convert the homogeneous version to an algebraic equation. The Fourier transform, however, is useful to convert from one space to another, such as time to frequency, space to k-space in solid state physics, or space to spatial frequency in optics. Here are the two most common forms of the transform and its inverse transform.

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt \; and \; f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\,d\omega,$$

where $f(t)\; and \; F(\omega)$ are Fourier transform pairs. Sometimes, it is convenient to write

$$F(\omega) = \mathcal{F}[f(t)] \text{ and } f(t) = \mathcal{F}^{-1}[F(\omega)].$$

The use of the factor $\sqrt{2\pi}$ is not standard. Some authors use 2π in only one of the factors, but the product must be 2π for one-dimensional transforms. In other applications, we might right

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}\,dx \text{ and } f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(k)e^{ikx}\,dk.$$

The Fourier transform of this equation is given by

$$-\omega^2 G(\omega, t') - 2i\,\gamma\omega G(\omega, t') + \omega_o^2 G(\omega, t') = \frac{1}{\sqrt{2\pi}} e^{i\omega t'}.$$

We solve for $G(\omega, t')$ to obtain

$$G(\omega, t') = \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega t'}}{\omega_o^2 - 2i\,\gamma\omega - \omega^2}.$$

Now we take the inverse transform to get

$$G(t, t') = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega t'}}{\omega_o^2 - 2i\,\gamma\omega - \omega^2} e^{-i\omega t} d\omega.$$

As you can see, this is a relative straightforward way to obtain the Green function in integral form, but we must use the residue theorem to do the integral. The integrand has two simple poles located at

$$-i\gamma \pm \sqrt{\omega_o^2 - \gamma^2}.$$

We have not yet done contour integration, but the result is given by

$$G(t, t') = 0 \quad t < t' \text{ and } G(t, t') = e^{-\gamma(t-t')} \frac{\sin[\Omega(\mathrm{t} - \mathrm{t}')]}{\Omega} \quad t > t'.$$

Keep in mind that we now have solved the problem for any $f(t)$. We need only solve the integral given by

$$x(t) = \int_0^t f(t') G(t, t') dt'$$