Physics 6303 Lecture 15 October 10, 2018

LAST TIME: Spherical harmonics and Bessel functions

Reminder of general solution in 3-dimensional cylindrical coordinates

$$V_{km}(\rho,\phi,z) = \sum_{k,m} \begin{Bmatrix} J_m(k\rho) \\ N_m(k\rho) \end{Bmatrix} \cdot \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix} \cdot \begin{Bmatrix} \sinh kz \\ \cosh kz \end{Bmatrix}$$

$$+ \sum_{k,m} \begin{Bmatrix} I_m(k\rho) \\ K_m(k\rho) \end{Bmatrix} \cdot \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix} \cdot \begin{Bmatrix} \sin kz \\ \cos kz \end{Bmatrix}$$

To make our job easier, here are some of the relevant properties of the different Bessel functions.

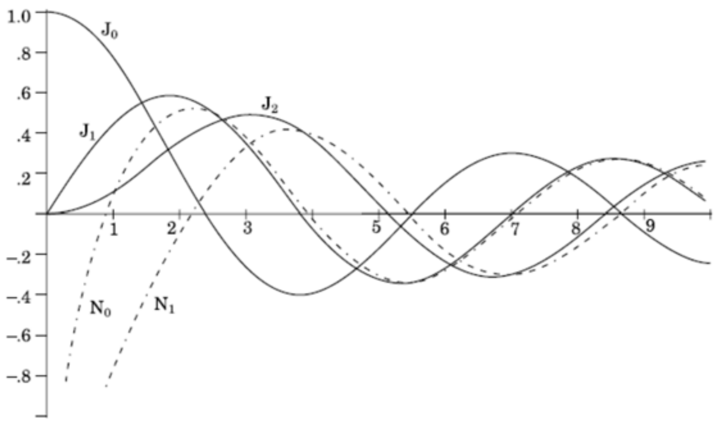
$J_m(k\rho)$ functions are well behaved everywhere, as they are oscillatory and appear somewhat like damped sine curves.

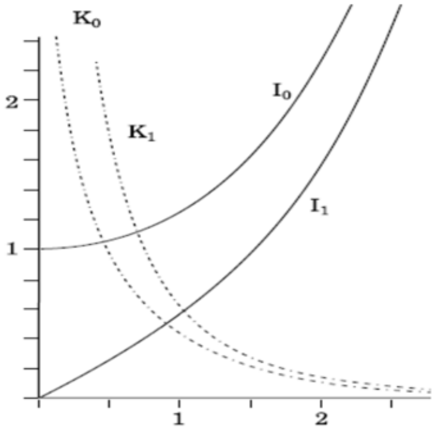
$N_m(k\rho)$ functions are also oscillatory, but they diverge at $\rho = 0,$ so they are excluded from the solution whenever the interior of a cylinder is involved.

$K_m(k\rho)$ functions are not oscillatory, but they diverge at $\rho = 0,$ so they, too, are excluded from the solution whenever the interior of a cylinder is involved.

$I_m(k\rho)$ functions are not oscillatory, but they diverge as $\rho \longrightarrow \infty,$ so they must be excluded from the solution whenever the exterior of the cylinder in involved.

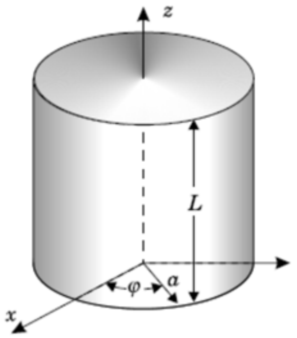
Here are the graphs of these functions to help you better keep track of their behavior.





You have to be careful if you are considering the annular region of a cylindrical geometry because none of the functions may be excluded arbitrarily.

Let’s do a couple of examples using Bessel functions.



Consider a right circular cylinder as shown with the potential on the bottom set to zero, the potential on the top surface set to $V_T(\rho, \phi, L)$, and the potential on the side set to $V_S(a, \phi, z)$. We have only one part set to zero, so we probably should solve this problem in a way similar to what we did when we had the Cartesian coordinates; *i.e.*, we need to make at least one equation with homogeneous boundary conditions. We hold the top surface at zero potential first, then we hold the curved surface at zero and superpose the two solutions.

Top and bottom grounded – curved side not.

Therefore, the BCs become $V(\rho, \phi, L) = V(\rho, \phi, 0) = 0$. We need to recall the properties of our functions to eliminate some of the constants. K_m and N_m cannot be in the solution because they are infinite at $\rho = 0$. Because we are solving the problem with the potential zero at the ends, the hyperbolic functions must also vanish. Furthermore, the cosine term must vanish and the eigenvalue is $\frac{n\pi}{L}$. Finally, we are left with the solution for $V_1(\rho, \phi, z)$ to be

$$V_1(\rho, \phi, z) = \sum_{m,n} (A_{mn} \sin m\phi + B_{mn} \cos m\phi)\, I_m\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right).$$

We evaluate this expression at the $\rho = a$ boundary to obtain

$$V_1(a, \phi, z) = \sum_{m,n} (A_{mn} \sin m\phi + B_{mn} \cos m\phi)\, I_m\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi z}{L}\right).$$

We use the orthogonality of the sines and cosines to get the values for A_{mn} and B_{mn}.

$$\begin{pmatrix} A_{mn} \\ B_{mn} \end{pmatrix} = \frac{2}{L\pi I_m\left(\frac{n\pi a}{L}\right)} \int_0^{2\pi} \int_0^{L} V_S(a, \phi, z) \begin{pmatrix} \sin m\phi \\ \cos m\phi \end{pmatrix} \sin\frac{n\pi z}{L}\, d\phi dz.$$

The column vectors are intended to be read in the same way as a ± sign. We cannot go further without specifying $V_S(a, \phi, z)$. Do notice, however, that $V_S(a, \phi, z)$ has a wide range of possible values because its dependence on both ϕ and z could be specified.

Now we ground the curved surface and leave the ends at the proper settings. This is designated as $V_2(\rho, \phi, z)$. After that, we superpose the solutions to obtain the complete solution. In mathematical terms, $V_S(a, \phi, z) = 0$. Once again, the interior cannot have K_m and N_m, and I_m can only be zero at the center, thereby preventing the BC from being satisfied. As a result of these restrictions, we are left with only

$$V_2(\rho,\phi,z) = \sum_m J_m(k\rho)\begin{pmatrix}\sin m\phi\\ \cos m\phi\end{pmatrix}\begin{pmatrix}\sinh kz\\ \cosh kz\end{pmatrix}.$$

We need the solution to vanish at $z = 0,$ so the cosh terms must go to zero. Finally, because the solution must vanish at $\rho = a,\ ka$ must be a root of J_m. Designate the i^{th} root J_m by ρ_{mi} so $k = \rho_{mi}a$. The general solution that satisfies the part of the boundary condition is given by

$$V_2(\rho,\phi,z) = \sum_{m,i} \sinh\frac{\rho_{mi}z}{a} J_m\left(\frac{\rho_{mi}\rho}{a}\right)(A_{mi}\sin m\phi + B_{mi}\cos m\phi).$$

At $z = L$, we obtain

$$V_T(\rho,\phi,L) = \sum_{m,i} \sinh\frac{\rho_{mi}L}{a} J_m\left(\frac{\rho_{mi}\rho}{a}\right)(A_{mi}\sin m\phi + B_{mi}\cos m\phi).$$

To shorten the writing, call $C_{mi} = A_{mi}\sinh\frac{\rho_{mi}L}{a}\ and\ D_{mi} = B_{mi}\sinh\frac{\rho_{mi}L}{a}$ to obtain

$$V_T(\rho,\phi,L) = \sum_{m,i} J_m\left(\frac{\rho_{mi}\rho}{a}\right)(C_{mi}\sin m\phi + D_{mi}\cos m\phi).$$

To find the values for C_{mi} and D_{mi}, we use the orthogonality condition for both sines, cosines, and the Bessel function by multiplying the equation above by

$$J_\ell\left(\frac{\rho_{\ell j}\rho}{a}\right)\begin{pmatrix}\cos\ell\phi\\ \sin\ell\phi\end{pmatrix}$$

and integrate over the top of the cylinder to obtain

$$\sum_{m,i}\int_0^a\int_0^{2\pi}(C_{mi}\sin m\phi + D_{mi}\cos m\phi)J_m\left(\frac{\rho_{mi}\rho}{a}\right)J_\ell\left(\frac{\rho_{\ell j}\rho}{a}\right)\begin{pmatrix}\cos\ell\phi\\ \sin\ell\phi\end{pmatrix}\rho d\rho d\phi$$

$$= \int_0^a\int_0^{2\pi} V_T(\rho,\phi,L)\, J_\ell\left(\frac{\rho_{\ell j}\rho}{a}\right)\begin{pmatrix}\cos\ell\phi\\ \sin\ell\phi\end{pmatrix}\rho d\rho d\phi.$$

We may use the orthogonality of the trignometric functions to obtain for the left hand side

$$LHS = \sum_i \begin{pmatrix}C_{\ell i}\\ D_{\ell i}\end{pmatrix}\pi\int_0^a J_m\left(\frac{\rho_{mi}\rho}{a}\right)J_\ell\left(\frac{\rho_{\ell j}\rho}{a}\right)\rho d\rho d\phi$$

Finally, we may use the orthogonality condition for the Bessel functions given by

$$\int_0^a J_m\left(\frac{\rho_{mi}\rho}{a}\right)J_\ell\left(\frac{\rho_{\ell j}\rho}{a}\right)\rho d\rho = \frac{1}{2}a^2 J_{\ell+1}(\rho_{\ell j})\delta_{ij}$$

to obtain

$$\begin{pmatrix} C_{\ell j} \\ D_{\ell j} \end{pmatrix} \frac{1}{2} \pi a^2 J_{\ell+1}(\rho_{\ell j}) = \int_0^a \int_0^{2\pi} V_T(\rho, \phi, L) J_\ell \left(\frac{\rho_{\ell j} \rho}{a} \right) \begin{pmatrix} \cos \ell\phi \\ \sin \ell\phi \end{pmatrix} \rho d\rho d\phi.$$

The original expansion coefficients are found to be

$$\begin{pmatrix} A_{\ell j} \\ B_{\ell j} \end{pmatrix} = \frac{2}{\pi a^2 J_{\ell+1}(\rho_{\ell j}) \sinh \frac{\rho_{\ell j} L}{a}} \int_0^a \int_0^{2\pi} V_T(\rho, \phi, L) J_\ell \left(\frac{\rho_{\ell j} \rho}{a} \right) \begin{pmatrix} \cos \ell\phi \\ \sin \ell\phi \end{pmatrix} \rho d\rho d\phi.$$

The complete solution is given by the sum of $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$. If we were to put different potentials on all three separate parts, we would have to superpose three solutions. You can certainly see that specification of the details of the BCs determines exactly how difficult the problems ae to finish.

One of the added difficulties using Bessel functions comes from the fact that if you are given a charge distribution to deal with, you will have to calculate the derivative of the radial part of the general solution, and that will involve taking the derivative of a Bessel function with respect to ρ.

Before we leave these BV problems, let me review the complete solutions for 2-D cylindrical coordinates and 3-D spherical coordinates. Here are the most general solutions for these geometries.

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell \frac{1}{r^{(\ell+1)}} \right) P_\ell(\mu)$$

$$V(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m} r^\ell + \frac{b_{\ell m}}{r^{\ell+1}} \right) P_\ell^m(\mu) e^{im\phi} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m} r^\ell + \frac{b_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)$$

INSERT LECTURE 15 A HERE PRELIMINARY TO GREEN FUNCTIONS.

One-dimensional Green functions.

Suppose we are given some physical system in a region bounded from 0 to L that is described by a differential equation. We have some input to that system that is described by $I(x)$ that may be represented by

$$I(x) = \int_0^L I(x') \delta(x - x') dx',$$

where $\delta(x - x')$ is the Dirac delta function whose main purpose is to serve as a sifting function so that

$$\int_{-\infty}^{\infty} f(x) \delta(x - a)\, dx = f(a).$$

It is important to understand that the delta function is not a function but the limit of a sequence of functions. Another useful property is given by

$$\delta(ax) = \frac{1}{|a|}\delta(x).$$

The fundamental idea of a Green function is that we solve our problem for a unit impulse function $\delta(x - x')$ and then superpose the solutions to get the solution to the actual input function. We call the response to the delta function a Green function, so this means that our solution $y(x)$ is given by

$$y(x) = \int_0^L I(x')G(x, x')dx',$$

where the Green function is symmetric in x and x'.
Let's look at a relatively simply example to cement the ideas.

Consider a particle moving under the influence of a driving force $F(t)$ with damping proportional to the velocity v. Our equation of motion is then given by

$$m\frac{dv}{dt} + kv = F(t).$$

To solve this problem using a Green function, we create the same differential equation using the Green function *G* and the Dirac delta function as the source to obtain

$$m\frac{dG}{dt} + kG = \delta(t - t').$$

If $t \neq t'$, the differential equation is homogeneous, $m\frac{dG}{dt} + kG = 0$, and is not hard to solve. We obtain

$$G = A\ \exp\left(-\frac{k}{m}t\right).$$

We must realize that the system was set into motion at some time $t = t'$, and that before it was set into motion, causality requires its response to be zero. This leads to the following solution for $G(t, t')$ given by

$$G(t, t') = \begin{cases} 0 & t < t' \\ A\ \exp\left(-\frac{k}{m}t\right) & t > t' \end{cases}$$

Essentially, the delta function at $t = t'$ serves to divide the region into two parts. This method is sometimes referred to as the division-of-region method. We need to determine A, and this is done by integrating the differential equation through the point $t = t'$. This process is usually carried out by taking the limits from $t' - \epsilon\ to\ t' + \epsilon$ and then letting ϵ go to zero.

Therefore,

$$\int_{t'-\epsilon}^{t'+\epsilon}\left(m\frac{dG}{dt}+kG\right)dt=\int_{t'-\epsilon}^{t'+\epsilon}\delta(t-t')dt=1.$$

Integrating the left hand side gives

$$mG|_{t'-\epsilon}^{t'+\epsilon}+k\int_{t'-\epsilon}^{t'+\epsilon}(G)dt=1$$

The integral is bounded and will become zero in the limit of ϵ going to zero. We evaluate the other term for ϵ going to zero to obtain

$$m\ \left[A\ \exp\left(-\frac{k}{m}t\right)-0\right]=1$$

so

$$A=\frac{1}{m}\exp\left(\frac{k}{m}t'\right).$$

Therefore, the Green function is given by

$$G(t,t')=\begin{cases}0 & t<t'\\ \dfrac{1}{m}\exp\left(-\dfrac{k}{m}[t-t']\right) & t>t'\end{cases}$$

Our solution for any $F(t)$ we wish to use is just given by

$$v(t)=\int_0^t F(t')G(t,t')dt'.$$

We may input a piecewise force function and carry out the integration. This is one of the nice features of a Green function – once you know it, you actually have the solution for may different input functions, no matter what they are.

If you are dealing with a problem that is a Sturm-Liouville equation, we can usually determine the Green function as an expansion of the eigenfunctions of the ODE. If the eigenfunctions have been normalized, we may write the orthogonality condition as

$$\int_a^b w(x)y_m(x)y_n(x)dx=\delta_{mn}.$$

To follow the same procedure as before, we insert the Green function into the Sturm-Liouville equation with the Dirac delta function as the unit source function. Once we get the Green function, we know that our solution is given by

$$y(x)=\int_0^L G(x,x')f(x')dx'.$$

Consider the general case first to obtain

$$\frac{d}{dx}\left(f(x)\frac{dG}{dx}\right) - g(x)G + \lambda w(x)G = -4\pi\delta(x-x').$$

We know we may expand *G* in terms of the eigenfunctions for this system to obtain

$$G(x,x') = \sum_{n=0}^{\infty} \beta_n\,(x')y_n(x)$$

We insert this expansion of $G(x,x')$ into the Sturm-Liouville equation to obtain

$$\sum_{n=0}^{\infty} \beta_n\,(x')\left[\frac{d}{dx}\left(f(x)\frac{d}{dx}y_n(x)\right) - g(x)y_n(x) + \lambda_n w(x)y_n(x)\right] = -4\pi\delta(x-x').$$

Every eigenfunction satisfies the Sturm-Liouville equation so we may replace the parts with the differentials with $\lambda w(x)y_n(x)$ to get

$$\sum_{n=0}^{\infty} \beta_n\,(x')[-\lambda_n w(x)y_n(x) + \lambda w(x)y_n(x)] = -4\pi\delta(x-x').$$

We multiply the equation by $y_m(x)$ and integrate over the range and find

$$\int_a^b \sum_{n=0}^{\infty} (\lambda - \lambda_n)\beta_n\,(x')w(x)[y_n(x)y_m(x)]\,dx = -4\pi\int_a^b \delta(x-x')y_m(x)dx$$

The integral is only over x so

$$\sum_{n=0}^{\infty} (\lambda - \lambda_n)\beta_n\,(x')\int_a^b w(x)[y_n(x)y_m(x)]dx = -4\pi y_m(x')$$

The integral on the LHS is just the orthogonality condition stated earlier which leads to

$$(\lambda - \lambda_m)\beta_m(x') = -4\pi y_m(x')$$

Finally,

$$\beta_m(x') = 4\pi\frac{y_m(x')}{(\lambda_m - \lambda)}$$

And our Green function becomes

$$G(x,x') = 4\pi\sum_{n=0}^{\infty}\frac{y_n(x')y_n(x)}{(\lambda_n - \lambda)}.$$

NEXT TIME: Examples and Green functions for multidimensional cases.