Physics 6303 Lecture 15C October 10, 2018

LAST TIME: Spherical harmonics and Bessel functions

Dirac delta function

To understand better the Green function development, it is helpful to delve briefly into the Dirac delta "function."(DDF) The word "function" is in quotation marks because the Dirac delta function is really not a function, but the limit of a sequence of functions that satisfy the following properties.

1. Its value is zero everywhere except where its argument is zero.
2. Its value is infinite where its argument is zero.
3. The integral of the function is 1.

Its primary use is to represent impulses, mass distributions such as points or rings, and charge distributions such as points or rings.

For example, the density of a point mass at the origin is written as

$$\rho(\mathbf{r}) = \mathrm{M}\delta(x)\delta(y)\delta(z).$$

Then, the integral of $\rho(\mathbf{r})dV$ is given by

$$M = M \iiint\limits_{-\infty}^{+\infty} \delta(x)\delta(y)\delta(z)dxdydz = M.$$

One of the most important properties is the so-called sifting property; *i.e.*,

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0) \text{ or } \int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a).$$

To prove the identities concerning the DDF, we usually use one of the representations of it. For example, the functions given by

$$\phi_n(x) = \frac{1}{n\pi}\frac{\sin^2 nx}{x^2}$$

are infinitely differentiable and also become more peaked with increasing n. Although not obvious without doing some contour integration, the following equation given by

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{1}{n\pi}\frac{\sin^2 nx}{x^2} f(x) = f(0).$$

This means that this function satisfies the sifting property of the DDF and is a good candidate to use for other proofs. On the following page, I list a few of the most important identities that are useful for our future work.

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$\int_{-\infty}^{+\infty} \delta'(x) f(x) dx = -f'(0)$$

$$\int_{-\infty}^{+\infty} \delta[g(x)] f(x) dx = \sum_{i=1}^{N} \frac{f(x_{oi})}{|g'(x_{oi})|} \text{ or } \delta[g(x)] = \sum_{i=1}^{N} \frac{\delta(x - x_{oi})}{|g'(x_{oi})|},$$

where the sum is over all the zeros of the function $g(x)$; $g(x_{oi}) = 0$. For example,

$$\delta(x^2 - a^2) = \frac{\delta(x-a)}{2|a|} + \frac{\delta(x+a)}{2|a|}.$$

Note that $x = \pm a$ and $g'(a) = 2a$ and $g'(-a) = -2a$.

You might recall that the DDF was used in the closure relationship that we derived for any general Sturm-Liouville system. This means that that are many representations of the DDF in terms of expansions of eigenfunctions. Here are a couple of very useful representations.

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx} dk \text{ and } \delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t} dk$$

Using the DDF to represent mass or charge distributions is relatively straightforward with Cartesian coordinates. This is ultimately because the metric tensor has no off-diagonal elements and has ones along the diagonal. In orthogonal curvilinear coordinates, however, the diagonal elements are not ones, so you have to be more careful about using the DDF. In spherical coordinates, the volume element is $dV = r^2 dr d(\cos\theta) d\phi$ so that the DDFZ must take the form given by

$$\delta(\mathbf{r} - \mathbf{a}) = r^{-2}\delta(r - a)\delta(\cos\theta - \cos\theta_a)\delta(\phi - \phi_a).$$

This form ensures that the DDF integrates to the correct value for the charge or mass distribution. In general, the DDF may be written as

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{|J(x_i, \eta_i)|}\delta(\eta_1 - \eta_1')\delta(\eta_2 - \eta_2')\delta(\eta_3 - \eta_3'),$$

where $|J(x_i, \eta_i)|$ is the magnitude of the Jacobian for the transformation.