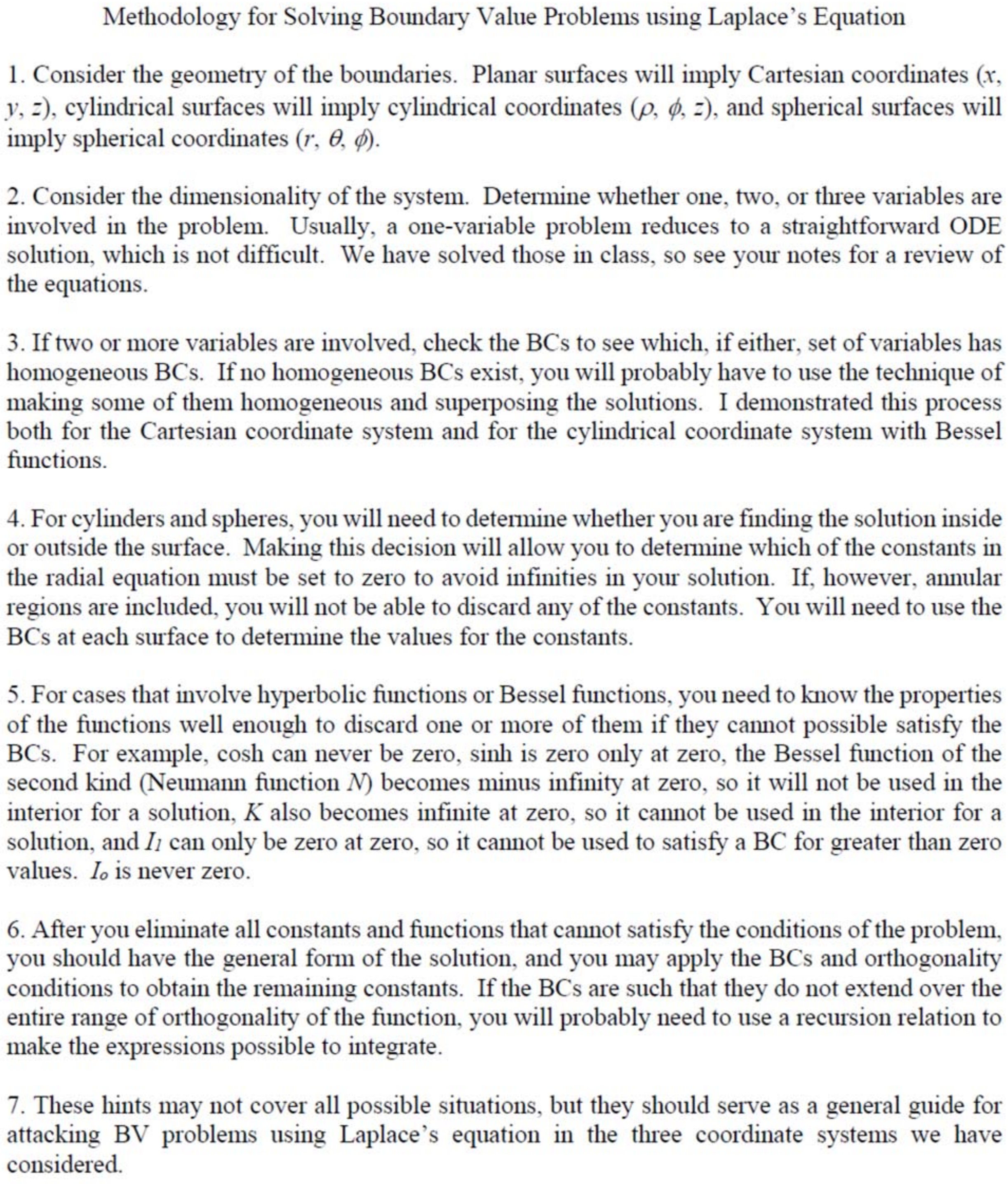
Physics 6303 Lecture 15B October 10, 2018

LAST TIME: Spherical harmonics and Bessel functions



Methodology for Solving Boundary Value Problems using Laplace's Equation

1. Consider the geometry of the boundaries. Planar surfaces will imply Cartesian coordinates (x, y, z), cylindrical surfaces will imply cylindrical coordinates (ρ, ϕ, z), and spherical surfaces will imply spherical coordinates (r, θ, ϕ).

2. Consider the dimensionality of the system. Determine whether one, two, or three variables are involved in the problem. Usually, a one-variable problem reduces to a straightforward ODE solution, which is not difficult. We have solved those in class, so see your notes for a review of the equations.

3. If two or more variables are involved, check the BCs to see which, if either, set of variables has homogeneous BCs. If no homogeneous BCs exist, you will probably have to use the technique of making some of them homogeneous and superposing the solutions. I demonstrated this process both for the Cartesian coordinate system and for the cylindrical coordinate system with Bessel functions.

4. For cylinders and spheres, you will need to determine whether you are finding the solution inside or outside the surface. Making this decision will allow you to determine which of the constants in the radial equation must be set to zero to avoid infinities in your solution. If, however, annular regions are included, you will not be able to discard any of the constants. You will need to use the BCs at each surface to determine the values for the constants.

5. For cases that involve hyperbolic functions or Bessel functions, you need to know the properties of the functions well enough to discard one or more of them if they cannot possible satisfy the BCs. For example, cosh can never be zero, sinh is zero only at zero, the Bessel function of the second kind (Neumann function N) becomes minus infinity at zero, so it will not be used in the interior for a solution, K also becomes infinite at zero, so it cannot be used in the interior for a solution, and I_1 can only be zero at zero, so it cannot be used to satisfy a BC for greater than zero values. I_o is never zero.

6. After you eliminate all constants and functions that cannot satisfy the conditions of the problem, you should have the general form of the solution, and you may apply the BCs and orthogonality conditions to obtain the remaining constants. If the BCs are such that they do not extend over the entire range of orthogonality of the function, you will probably need to use a recursion relation to make the expressions possible to integrate.

7. These hints may not cover all possible situations, but they should serve as a general guide for attacking BV problems using Laplace's equation in the three coordinate systems we have considered.

Before we launch into the topic of Green functions, let me give you a brief overview of why they are important and what they allow us to do.

A Green function is used when we have an ODE that is nonhomogeneous (or inhomogeneous); *i.e.*; there is a source term on the RHS. A damped, driven harmonic oscillator is a convenient example that you have probably seen before. The equation is given by

$$m\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+kx(t)=F(t).$$

A Green function G uses the linearity of the ODE to determine the response of the ODE solution to an impulse function (Dirac delta "function") and then superposes the solutions for the impulse function to get the solution for the function, here, $F(t)$. In this case,

$$m\frac{d^2G(t,t')}{dt^2}+b\frac{dG(t,t')}{dt}+kG(t,t')=\delta(t-t').$$

Let's suppose we think of our driving function $F(t)$ as $F(t)=\int F(t')\ \delta(t-t')dt'$. We would then use superposition to write

$$x(t)=\int F(t')\,G(t,t')dt'.$$

The same argument holds for a physical system bounded in space with boundary conditions. We have seen many of these, but we only considered Laplace's equation for that. The analogous equation for electrostatics is Poisson's equation, which is just Laplace's equation with a charge distribution added to the RHS. The real value in the Green function approach is that once we find the Green function for a particular differential equation and set of boundary conditions or initial conditions, we are then able to find the solution to the differential equation for any source term by simply doing the integral above. The problem is usually finding the Green function. There are at least three methods commonly used to find the Green function.

1. Division of region – The delta function conveniently divides the region into t < t' and t > t'. We then integrate the differential equation across the t = t' region to obtain information about the constants.

2. Much of our work has dealt with Sturm – Liouville ODEs, so we know solutions may be represented in terms of the eigenfunctions. In these cases, our Green function may be represented in terms of an eigenfunction expansion.

3. We may use Fourier transforms to convert our ODE equation to an algebraic equation, solve for $G(\omega,t')$ and then invert the result to get $G(t,t')$. Unfortunately, for us at this time, we usually need to do a contour integration using the residue theorem that we have not yet studied. I will defer this method until we cover the necessary material.

Let's look at a detailed one-dimensional example to fix the ideas.

Now to