LAST TIME: Associated Legendre polynomials, spherical harmonics, and examples.

One more example using spherical harmonics.

Determine the potential inside a conducting spherical shell having $V(R,\theta,\phi) = V_o$ for $0 \le \phi < \pi$ and 0 for $\pi \le \phi < 2\pi$. We know immediately the correct form of the solution must be given by

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m} r^{\ell}\right) Y_{\ell m}(\theta,\phi)$$

because the $b_{\ell m}$ must be zero to keep the potential finite inside the sphere. Apply the boundary conditions to obtain

$$V(R,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m} R^{\ell}\right) Y_{\ell m}(\theta,\phi) = V_o \quad 0 \le \phi < \pi$$

and 0 for $\pi \le \phi < 2\pi$. You should notice once again that the boundary condition is not the same over the full range of the orthogonality condition. Why is this not a problem for us this time?

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m} R^{\ell}\right) \int Y_{\ell m}(\theta,\phi)\, Y^{*}_{\ell' m'}(\theta,\phi) d\Omega = V_o \int_{-1}^{1} \int_{0}^{\pi} Y^{*}_{\ell' m'}(\theta,\phi) d\mu\, d\phi.$$

The LHS will only have a nonzero value when $\ell = \ell'$ and $m = m'$. Its value is one so that

$$a_{\ell m} R^{\ell} = V_o \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \int_{-1}^{1} P_{\ell}^{m}(\mu) d\mu \left(\frac{1-(-1)^m}{im}\right).$$

I split apart the spherical harmonic term and did the integral over phi. The last term in parenthesis is zero unless *m* is odd. This essentially solves the problem except for a few points that need to be considered. $m = \ell = 0$ is special because the spherical harmonic is constant. The second point is that for $m = 0, \ell \neq 0$ the result is also zero. Do you see why this is true? Here are the comments.

$$\int_{-1}^{+1} \int_{0}^{\pi} Y_{00} = \frac{1}{\sqrt{4\pi}} (2)(\pi) = \sqrt{\pi} \Rightarrow a_{00} = \frac{V_0}{2}.$$

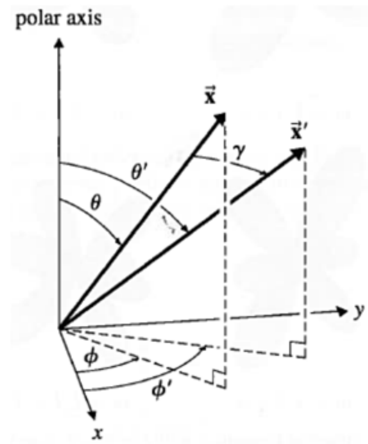
For the case m = 1, $\ell \neq 0$ $\int_{-1}^{+1} P_0(\mu)\, P_{\ell}(\mu) d\mu = 0.$

For any one of the problems we have encountered so far, we may have to consider the following situations.

1. Specified potential over only part of the surface so that the orthogonality condition must be handled carefully.

2. Specified surface charge density over only part of the surface so that the orthogonality condition must be handled carefully. Use electric field here – derivative of the potential. You will have to solve the problem inside and outside the region because of the discontinuity in the electric field.

3. Calculating the potential on the axis of a given charge density and then matching the "boundary conditions" on the axis.



You are now equipped to handle a wide variety of BV problems, tedious though they may be. The addition theorem for spherical harmonics allows us to express the Legendre polynomials in terms of the spherical harmonics. Consider the figure to the right. The addition theorem is given by the expression

$$P_\ell(\cos\gamma) = \sum_{m=-\ell}^{m=\ell} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta,\phi) Y^*_{\ell m}(\theta',\phi').$$

Notice that, since $P_1(\cos\gamma) = \cos\gamma$, it is also possible to write $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$ by using ordinary vector analysis. Recall that I showed you the multipole expansion using Cartesian coordinates. You can now see that the multipole expansion may be carried out using spherical harmonics and is even more general. From our earlier work,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma).$$

If we have a number of charges inside a region, we may use the addition theorem to write a very general multipole expansion as

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\sum_i q_i \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1}\frac{r_i^\ell}{r^{\ell+1}} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta_i,\phi_i) Y^*_{\ell m}(\theta,\phi)$$

$$= \frac{1}{4\pi\epsilon_o}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\sum_i q_i r_i^\ell\, Y_{\ell m}(\theta_i,\phi_i)\frac{Y^*_{\ell m}(\theta,\phi)}{r^{\ell+1}}.$$

The complex conjugate part may be interchanged, depending on the text. We have isolated the terms that involve the charge to obtain

$$M_{\ell m} = \frac{4\pi}{2\ell+1}\sum_i q_i r_i^\ell\, Y_{\ell m}(\theta_i,\phi_i)$$

Finally,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} M_{\ell m}\frac{Y_{\ell m}^*(\theta,\phi)}{r^{\ell+1}}.$$

These are the multipole moments written in terms of the spherical harmonics. You will find spherical harmonics useful in angular momentum considerations in quantum mechanics.

Let's return to the cylindrical symmetry case where the *z*-direction is included. Recall

$$\nabla^2 V(\rho,\phi,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

We separate the variables by assuming $V(\rho,\phi,z) = R(\rho)\Phi(\phi)Z(z)$.
Carrying out the same process we have done several times yields

$$\frac{1}{R\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\Phi\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0.$$

The z-term is nicely separated, and we have no reason to think that it should be oscillatory. We pick the separation constant to be positive so that

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = +k^2$$

and the solutions are $Z = e^{\pm kz}$. The remaining equation becomes

$$\frac{1}{R\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\Phi\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + k^2 = 0.$$

If we multiply by ρ^2, the $\phi -$ term is also separated, and we obtain

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} + \rho^2 k^2 = 0.$$

Therefore,

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2,$$

Where we choose $-m^2$ because we need a periodic function in ϕ. Now we are left with the equation in ρ as

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \rho k^2 R - \frac{m^2}{\rho}R = 0.$$

This is written in the form to indicate a Sturm-Liouville ODE with $f(\rho) = \rho, g(\rho) = \frac{m^2}{\rho}, \lambda = k^2$, and $w(\rho) = \rho$. We are assured that our solutions will give us an complete set of orthogonal functions. It is common to make a substitution to write the equation in terms of the dimensionless variable $x = k\rho$ so that the equation becomes

$$\frac{d}{dx}\left(x\frac{dR}{dx}\right) + xR - \frac{m^2}{x}R = 0.$$

There is a regular singular point at $x = 0$, so we use care in solving the equation. The method of Frobenius is designed just for this case because it builds in a second term that allows for fractional and/or inverse powers in the series. We assume a solution of the form

$$R = \sum_{n=0}^{\infty} a_n\, x^{n+p} \text{ (note the additional power of } p\text{)}$$

and substitute this expression back into the ODE to obtain the result

$$\sum_{n=0}^{\infty}(n+p)(n+p-1)\, a_n x^{n+p-1} + \sum_{n=0}^{\infty}(n+p)\, a_n x^{n+p-1} + \sum_{n=0}^{\infty} a_n x^{n+p+1}$$

$$-m^2 \sum_{n=0}^{\infty}(n+p)\, a_n x^{n+p-1} = 0.$$

We set *n* = 0 and look at the coefficients of x^{p-1} to obtain the indicial equation for *p*. This yields

$$p(p-1) + p - m^2 = 0 \text{ and } p = \pm m.$$

The positive value of $p = m$ gives an analytic solution, whereas the negative value does not. The negative value still leaves a value of x^{-1}. We will return to this point in a while. Changing n to k, we look at the $k + p - 1$ power of *x* to obtain

$$(k+p)(k+p-1)a_k + (k+p)a_k + a_{k-2} - m^2 a_k = 0.$$

We solve for a_k in terms of a_{k-2} to obtain

$$a_k = -\frac{a_{k-2}}{(k+p)^2 - m^2} = -\frac{a_{k-2}}{(k^2 + 2pk + p^2) - m^2} = -\frac{a_{k-2}}{k(k \pm 2m)}$$

Now let's consider the positive case. Therefore,

$$a_k = -\frac{a_{k-2}}{k(k+2m)} \Rightarrow a_{2n} = -\frac{a_{2n-2}}{2n(2n+2m)} = -\frac{a_{2n-2}}{2^2 n(n+m)}.$$

Notice that I have used the fact that $a_1 = 0$ because in the term for $n = 1$, no other power exists.

In order to recognize these terms in a more conventional way, we need to digress a bit to mention the gamma function. The gamma function was originally defined for real *x* as

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt \;\Rightarrow \Gamma(1) = \int_0^\infty e^{-t}\, dt = 1.$$

If we integrate the definition by parts as follows, it is possible to obtain a recursion relation for the gamma function.

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt = t^{x-1}(e^{-t})|_0^\infty - \int_0^\infty (x-1)(-e^{-t})\, t^{x-2} dt$$

yielding

$$\Gamma(x) = (x-1)\Gamma(x-1).$$

Therefore, if x is an integer

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)(n-2)\cdots 1\Gamma(1) = n!$$

We may convert the recursion relation to a form that involves the gamma function, thereby giving

$$a_2 = -\frac{a_o}{2^2(p+1)} = \frac{a_o\Gamma(p+1)}{2^2\Gamma(p+2)}.$$

If we take $a_o = \frac{1}{2^m\Gamma(m+1)}$, we may finally write

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{m+2n}.$$

Rather than write the entire solution for the –*m* case, I will simply state the results in the interest of getting into the solutions to a few potential problems. The –m case does not produce a linearly independent solution, but rather is related to $J_m(x)$ by the relationship

$$J_{-m}(x) = (-1)^m J_m(x).$$

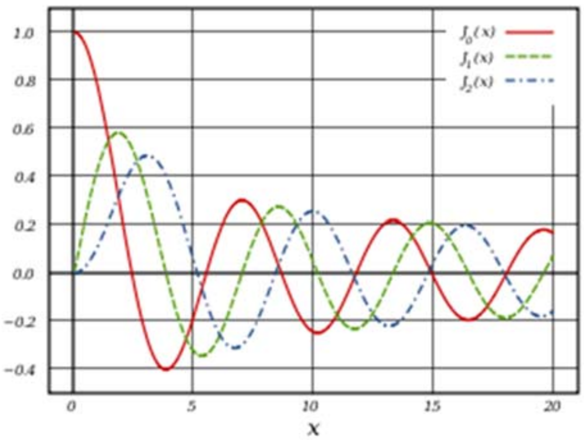
So, what is the second linearly independent solution to Bessel's equation? Without proof, I will state that the second solution is known as the Neumann function (or Bessel functions of the second kind) given by

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}.$$

There are two additional functions that come from the Bessel functions. They are the Hankel functions of the first and second kind given by

$$H_m^{(1)}(x) = J_m(x) + iN_m(x) \text{ and } H_m^{(2)}(x) = J_m(x) - iN_m(x).$$

For the record, you will sometimes see these types of functions referred to as cylindrical Bessel functions to distinguish them from spherical Bessel functions. Occasionally, $N_m(x)$ is designated as $Y_m(x)$. Here are some graphs of the first few orders of Bessel functions. Some of you might remember that the $J_o(x)$ is the one that occurs in circular diffraction problems. It is the one from which the Rayleigh criterion for resolution is derived. There are numerous properties of Bessel functions that are useful to know about before we start to apply them to potential problems. For small arguments of *x*



$$J_m(x) \cong \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m.$$

The Neumann functions are not well behaved at $x = 0$ because they have a logarithmic singularity given by

$$N_o(x) \approx \frac{2}{\pi}\ln x$$

And because for $m > 0, N_m(x)$ diverges as an inverse power of *x*.

$$N_m(x) \approx -\frac{(m-1)!}{\pi}\left(\frac{2}{x}\right)^m$$

For large values of x, both $J_m(x)$ and $N_m(x)$ both oscillate like damped sine functions given by

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}}\cos\ (x - \frac{m\pi}{2} - \frac{\pi}{4})$$

and

$$N_m(x) \approx \sqrt{\frac{2}{\pi x}}\sin\ (x - \frac{m\pi}{2} - \frac{\pi}{4})$$

For large x, the Hankel functions are like complex exponentials

$$H_m^{(1,2)} = \sqrt{\frac{2}{\pi x}}\exp\left[\pm i\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right].$$

Some of the recurrence relationships are given on pages 654 and 646 of your textbook. I will not repeat them here. Bessel functions find uses in many areas of physics including diffraction theory, vibrating drum heads, cylindrical potential well (cylindrical bessel functions), and spherical potential well (spherical Bessel functions). Because they are solutions to a Sturm-Liouville ODE, they are also a complete set of orthogonal functions. In order to get on with some examples of potential problems, I state without proof the following additional relationships.

$$e^{ik\rho\sin\phi} = \sum_{m=-\infty}^{\infty} J_m(k\rho)e^{im\phi}$$

as the generating function and an integral representation given by

$$J_m(kr) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikr\sin\phi - im\phi}\, d\phi.$$

The orthogonality conditions are a little more complicated and are given by

$$\int_0^a \rho\, J_m(k_{mn'}\rho)J_m(k_{mn}\rho)d\rho = 0 \;\; for\; n \neq n'$$

and

$$\int_0^a \rho\, J_m(k_{mn'}\rho)J_m(k_{mn}\rho)d\rho = 0$$

$$\int_0^a \rho\; [J_m(k_{mn}\rho)]^2 d\rho = \frac{a^2}{2}[J'_m(k_{mn}a)]^2.$$

There is one last issue about Bessel functions, and that concerns our choice for the sign of the separation constant k^2. Had we chosen the negative sign, our equation for Z would have the following solutions given by

$$Z = a\sin kz + b\cos kz,$$

and the ordinary Bessel equation becomes

$$\frac{d}{dx}\left(x\frac{dR}{dx}\right) - xR - \frac{m^2}{x}R = 0,$$

which is called the modified Bessel equation. Not surprising, it has the solutions that are modified Bessel functions given by

$$I_m(x) = i^{-m}J_m(ix)$$

The second solution is usually designated by

$$K_m(x) = \frac{\pi}{2}i^{m+1}H_m^{(1)}(ix).$$

This means that the complete solution for the cylindrical geometry is given by

$$V_{\lambda m}(r,\varphi,z)=\sum_{\lambda,m}\left\{\begin{array}{c}\mathrm{J}_m(\lambda r)\\ \mathrm{N}_m(\lambda r)\end{array}\right\}\cdot\left\{\begin{array}{c}\sin m\varphi\\ \cos m\varphi\end{array}\right\}\cdot\left\{\begin{array}{c}\sinh\lambda z\\ \cosh\lambda z\end{array}\right\}$$

$$+\sum_{\lambda,m}\left\{\begin{array}{c}\mathrm{I}_m(\lambda r)\\ \mathrm{K}_m(\lambda r)\end{array}\right\}\cdot\left\{\begin{array}{c}\sin m\varphi\\ \cos m\varphi\end{array}\right\}\cdot\left\{\begin{array}{c}\sin\lambda z\\ \cos\lambda z\end{array}\right\}$$

To use this form of the solution, we must know which functions are well behaved at the origin and which are well behaved at infinity.

NEXT TIME: Bessel function examples and Green functions.