Lecture 12 was examination 1.

LAST TIME: Cylindrical coordinates, spherical coordinates, and Legendre's equation

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{(\ell+1)}}\right)P_\ell(\mu).$$

Consider problems that are not axisymmetric; *i.e.*, the potential depends on ϕ. Solutions to equations of this type lead to spherical harmonics. Recall that

$$\nabla^2 V(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right) = 0$$

is Laplace's equation in spherical coordinates. This time, however, we will use

$$V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

as the separation variables. Substituting and simplifying, we obtain

$$\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\frac{1}{\Theta} + \frac{1}{\Phi r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

To isolate the Φ terms, we multiply by $r^2\sin^2\theta$. This leaves the last term as

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2.$$

Here again, we choose the negative sign because we know that this term has to be periodic 2π since you must get the same value when you go around the circle once time. We may either write the solution to this equation as sines and cosines or a complex exponential. Therefore,

$$\Phi \sim \sin m\phi \text{ and } \cos m\phi \text{ or } e^{\pm im\phi}.$$

The equation in r and θ becomes

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\frac{1}{\Theta} - m^2 = 0.$$

We now divide by $\sin^2\theta$ to isolate the r- equation from the $\theta -$ equation. This gives

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\frac{1}{\Theta} - \frac{m^2}{\sin^2\theta} = 0$$

so that

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = k$$

and

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\frac{1}{\Theta} - \frac{m^2}{\sin^2\theta} + k = 0$$

are the separated equations in r and θ. Once again, it is convenient to introduce $\mu = \cos\theta$ to obtain

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP}{d\mu}\right] - \frac{m^2}{(1-\mu^2)}P + \ell(\ell+1)P = 0,$$

where I switched from Θ to P and used $k = \ell(\ell+1)$. Just to keep up my theme, you can see that this is a Sturm-Liouville ODE with $f(\mu) = 1-\mu^2, g(\mu) = \frac{m^2}{(1-\mu^2)}, w(\mu) = 1,$ and $\lambda = \ell(\ell+1)$. The solutions to this equation are the associated Legendre functions designated by $P_\ell^m(\mu)$. We may do a series solution to this equation just as we have done before, but it is also possible to relate these functions to the polynomials we already found. We expand the derivative in Legendre's original equation and then continue to take additional derivatives m times. Here are the results of such action.

$$(1-\mu^2)\frac{d^2P_\ell}{d\mu^2} - 2\mu\frac{dP_\ell}{d\mu} + \ell(\ell+1)P_\ell = 0$$

$$(1-\mu^2)\frac{d^3P_\ell}{d\mu^3} - 2\times 2\mu\frac{d^2P_\ell}{d\mu^2} - 2\frac{dP_\ell}{d\mu} + \ell(\ell+1)\frac{dP_\ell}{d\mu} = 0$$

$$(1-\mu^2)\frac{d^4P_\ell}{d\mu^4} - 2\times 3\mu\frac{d^3P_\ell}{d\mu^3} + [\ell(\ell+1) - 2\times 3]\frac{d^2P_\ell}{d\mu^2} = 0$$

$$(1-\mu^2)\frac{d^{m+2}P_\ell}{d\mu^{m+2}} - 2(m+1)\mu\frac{d^{m+1}P_\ell}{d\mu^{m+1}} + [\ell(\ell+1) - m(m+1)]\frac{d^mP_\ell}{d\mu^m} = 0$$

We see that the m^{th} derivative of P_ℓ satisfies the equation

$$(1-\mu^2)y_m'' - 2(m+1)\mu y_m' + [\ell(\ell+1) - m(m+1)]y_m = 0.$$

Make the substitution $y_m = (1-\mu^2)^{-\frac{m}{2}}z(\mu)$ to obtain

$$(1-\mu^2)z'' - 2\mu z' + z\left[\ell(\ell+1) - \frac{m^2}{(1-\mu^2)}\right] = 0.$$

Because z satisfies the associated Legendre equation, $Cz(\mu) = P_\ell^m(\mu)$. If we choose $C = (-1)^m$, Then we may write

$$P_\ell^m(\mu) = (-1)^m (1-\mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} P_\ell(\mu).$$

The orthogonal condition may be written as

$$I_{\ell m} = \int_{-1}^{1} P_\ell^m(\mu) P_\ell^m(\mu)\, d\mu = \int_{-1}^{1} (1-\mu^2)^m \frac{d^m P_\ell(\mu)}{d\mu^m} \frac{d^m P_\ell(\mu)}{d\mu^m}\, d\mu.$$

I will only show part of the method. We have to integrate by parts – this is an expected method when two derivatives are involved. Therefore,

$$I_{\ell m} = (1-\mu^2)^m \frac{d^m P_\ell(\mu)}{d\mu^m} \frac{d^{m-1} P_\ell(\mu)}{d\mu^{m-1}} \Bigg|_{-1}^{1}$$

$$- \int_{-1}^{1} \frac{d}{d\mu}\left[(1-\mu^2)^m \frac{d^m P_\ell(\mu)}{d\mu^m}\right] \frac{d^{m-1} P_\ell(\mu)}{d\mu^{m-1}}\, d\mu.$$

The integrated term is zero (why?). The derivative must be expanded, and then the equation for the derivatives of $P_\ell(\mu)$ is used to obtain an expression as follows.

$$I_{\ell m} = (\ell + m)(\ell - m + 1) I_{\ell, m-1}.$$

Finally,

$$I_{\ell m} = \frac{(\ell+m)!}{(\ell-m)!} \frac{2}{2\ell+1}.$$

This means that the orthogonality condition is given by

$$\int_{-1}^{1} P_\ell^m(\mu) P_{\ell'}^m(\mu)\, d\mu = \frac{(\ell+m)!}{(\ell-m)!} \frac{2}{2\ell+1} \delta_{\ell\ell'}.$$

The complete solution for Laplace's equation in spherical coordinates is given by

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m} r^\ell + \frac{b_{\ell m}}{r^{\ell+1}}\right) P_\ell^m(\mu) e^{im\phi}.$$

It is customary to define the combination

$$\sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\mu) e^{im\phi} = Y_{\ell m}(\theta,\phi).$$

Finally,

$$\int_{-1}^{1}\int_{0}^{2\pi} Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi)\,d\phi d\mu = \delta_{\ell\ell'}\delta_{mm'} = \int Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi)d\Omega,$$

where the integral with respect to Ω is done over a sphere. For the record, some books prefer to write the spherical harmonics as $Y_\ell^m(\theta,\phi)$ to stay consistent with the P_ℓ^m. Please see your text on pages 759 and 760 for figures and values of different spherical harmonics. Obviously, these functions have wide use in quantum mechanics and everywhere else involving spherical symmetry.

Boundary value problems using spherical harmonics.

Determine the potential inside a hollow, conducting sphere or radius R with the potential on the surface specified to be $V_o(R,\theta,\phi)$. We know that $b_{\ell m}$ must be zero to keep the potential inside finite. Therefore,

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(a_{\ell m}r^{\ell}\right)Y_{\ell m}(\theta,\phi).$$

and

$$V_o(R,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(a_{\ell m}R^{\ell}\right)Y_{\ell m}(\theta,\phi)$$

Multiplying by $Y^*_{\ell' m'}(\theta,\phi)$ and using the orthogonality condition yields

$$\int_{0}^{2\pi}\int_{0}^{\pi} V_o(R,\theta,\phi)\,Y^*_{\ell' m'}(\theta,\phi)\sin\theta d\theta d\phi = a_{\ell' m'}R^{\ell'}.$$

Of course, we cannot proceed further without specifying $V_o(R,\theta,\phi)$.

Determine the potential inside a conducting spherical shell having $V(R,\theta,\phi) = V_o$ for $0 \le \phi < \pi$ and 0 for $\pi \le \phi < 2\pi$. We know immediately the correct form of the solution must be given by

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(a_{\ell m}r^{\ell}\right)Y_{\ell m}(\theta,\phi)$$

because the $b_{\ell m}$ must be zero to keep the potential finite inside the sphere. Apply the boundary conditions to obtain

$$V(R,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(a_{\ell m}R^{\ell}\right)Y_{\ell m}(\theta,\phi) = V_o \quad 0 \le \phi < \pi$$

and 0 for $\pi \le \phi < 2\pi$. You should notice once again that the boundary condition is not the same over the full range of the orthogonality condition. Why is this not a problem for us this time?

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(a_{\ell m}R^{\ell}\right)\int Y_{\ell m}(\theta,\phi)\,Y^*_{\ell' m'}(\theta,\phi)d\Omega = V_o\int_{-1}^{1}\int_{0}^{\pi}Y^*_{\ell' m'}(\theta,\phi)d\mu\, d\phi.$$

The LHS will only have a nonzero value when $\ell = \ell'$ and $m = m'$. Its value is one so that

$$a_{\ell m}R^{\ell} = V_o\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\int_{-1}^{1} P_\ell^m(\mu)d\mu\left(\frac{1-(-1)^m}{im}\right).$$

I split apart the spherical harmonic term and did the integral over phi. The last term in parenthesis is zero unless m is odd. This essentially solves the problem except for a few points that need to be considered. $m = \ell = 0$ is special because the spherical harmonic is constant. The second point is that for $m = 0, \ell \neq 0$ the result is also zero. Do you see why this is true?

NEXT TIME: More examples, addition theorem, and Bessel functions.