LAST TIME: Cylindrical coordinates, spherical coordinates, and Legendre's equation

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{(\ell+1)}}\right)P_\ell(\mu).$$

Consider problems that are no axisymmetric; i.e., the potential depends on $\phi.$ Solutions to equations of this type lead to spherical harmonics

Variable Coefficient, Second Order ODEs

Consider the ODE given by

$$b_2(x)y'' + b_1(x)y' + b_o(x)y = 0.$$

It is convenient to write the equation in a more standard form given by

$$y'' + P(x)y' + Q(x)y = 0,$$

where

$$P(x) = \frac{b_1(x)}{b_2(x)} \text{ and } Q(x) = \frac{b_o(x)}{b_2(x)}.$$

Because many of our equations in physics are represented by a power series, we need to know under what conditions our functions are analytic; i.e., they can be expanded in a Taylor series about x_o. x_o is an ordinary if both $P(x)$ and $Q(x)$ are analytic at x_o. x_o is a regular singular point if

$$(x - x_o)P(x) \text{ and } (x - x_o)^2\, Q(x) \text{ are analytic at } x_o.$$

Points that are not regular singular points are irregular. A couple of examples will serve as examples. Consider the differential equation given by $y'' - xy' + 2y = 0$. Here, $P(x) = -x$ and $Q(x) = 2$. Both $P(x)$ and $Q(x)$ are analytic so every value of x is analytic. Look at $2x^2y'' + 7x(x+1)y' - 3y = 0$. Consider

$$P(x) = \frac{7x(x+1)}{2x^2} \; and \; Q(x) = \frac{-3}{2x^2}.$$

Neither function is analytic at $x = 0,$ so $x = 0$ is a singular point. $xP(x) = \frac{7}{2}(x + 1)$ and is analytic at $x = 0$. Similarly, $x^2Q(x) = -\frac{3}{2}$ and $x = 0$ is a regular singular point. What is the significance of being a regular point?

When dealing with Legendre polynomials, there are instances when it is useful to have relationships between the various orders of the polynomials and/or between the polynomials and their derivatives. These relationships are called recursion relations and are determined usually by differentiating the generating function with respect to *x* or μ. Here are a few of the ones that find some usefulness on occasion.

$$\ell P_{\ell-1}(\mu) - (2\ell+1)\mu P_\ell(\mu) + (\ell+1)P_{\ell+1}(\mu) = 0,$$

which is said to be a pure recursion relation because it relates orders above and below $P_\ell(\mu)$ and has no derivatives. Here are some others

$$P_\ell(\mu) = P'_{\ell-1}(\mu) - 2\mu P'_{\ell-1}(\mu) + P'_{\ell+1}(\mu)$$

$$\ell P_\ell(\mu) = \mu P'_\ell(\mu) - P'_{\ell-1}(\mu)$$

$$(2\ell+1)P_\ell(\mu) = P'_{\ell+1}(\mu) - P'_{\ell-1}(\mu) \ast\ast$$

$$P_{\ell-1}(\mu) = \mu P_\ell(\mu) + \left(\frac{1-\mu^2}{\ell}\right)P'_\ell(\mu)$$

$$P_{\ell+1}(\mu) = \mu P_\ell(\mu) - \left(\frac{1-\mu^2}{\ell+1}\right)P'_\ell(\mu)$$

Before we work some detailed examples, it is worthwhile to consider the general types of problems that we can solve and to see what some of the problems we encounter are.

1. There are the very traditional problems where a grounded, conducting sphere is placed in an otherwise uniform electric field, and we are asked to determine the potential everywhere outside the sphere. Here,

Example 1: A hollow conducting sphere of radius R is divided into two halves at the equator by a thin insulating ring. The top half of the sphere is held at a potential V_o, whereas the bottom half is grounded (zero potential). Determine the potential everywhere.

Solution: We know that

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{(\ell+1)}}\right)P_\ell(\mu).$$

Consider the solution inside the sphere first. We know that $B_\ell = 0$ to keep the potential finite at the center. We apply the boundary conditions at the surface to obtain

$$V(R,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell R^\ell\right)P_\ell(\mu) = V_o \; for \; 1 \geq \mu > 0$$

$$= 0 \; for \; 0 > \mu \geq -1.$$

Because $\mu = \cos\theta$, this limit is really from 0 to $\frac{\pi}{2}$ and from $\frac{\pi}{2}$ to π. Now apply the orthogonality condition by multiplying each side by $P_{\ell'}(\mu)$ and integrating from -1 to 1 to obtain.

$$\int_{-1}^{1}\sum_{\ell=0}^{\infty}\left(A_\ell R^\ell\right)P_\ell(\mu)P_{\ell'}(\mu)d\,\mu = \int_{0}^{1}V_o\,P_{\ell'}(\mu)d\mu$$

The left hand side is no problem, but the right hand side does not involve two polynomials, nor does it extend over the entire range of the orthogonality condition. Therefore, we obtain

$$\left(A_\ell R^\ell\right)\frac{2}{2\ell+1} = V_o\int_0^1 P_\ell(\mu)d\mu.$$

If we could write the right hand side as an exact differential we would have the problem solved. Fortunately, one of the recursion relations gives us a way to do just that. The double-starred one is used to write

$$V_o\int_0^1 P_\ell(\mu)d\mu = \frac{V_o}{2\ell+1}\int_0^1\left[P'_{\ell+1}(\mu) - P'_{\ell-1}(\mu)\right]d\mu.$$

Therefore,

$$V_o\int_0^1 P_\ell(\mu)d\mu = \frac{V_o}{2\ell+1}\left[P_{\ell+1}(\mu) - P_{\ell-1}(\mu)\right]\Big|_0^1$$

Now, recall that $P_\ell(1) = 1$ for all ℓ so we obtain

$$V_o\int_0^1 P_\ell(\mu)d\mu = \frac{V_o}{2\ell+1}\left[-P_{\ell+1}(0) + P_{\ell-1}(0)\right].$$

We use the pure recursion relation to put $P_{\ell-1}(0)$ in terms of $P_{\ell+1}(0)$. Note that the middle term vanishes because $\mu = 0$.

Therefore,

$$V_o\int_0^1 P_\ell(\mu)d\mu = -\frac{V_o}{2\ell+1}P_{\ell+1}(0)\left(1+\frac{\ell+1}{\ell}\right) = -\frac{P_{\ell+1}(0)}{\ell}.$$

Finally,

$$A_\ell = -\frac{V_o}{2}\frac{2\ell+1}{\ell R^\ell}P_{\ell+1}(0);\quad \ell \geq 1.$$

To complete the problem, we need to obtain the results for $\ell = 0$. Use $P_o(\mu) = 1$. Then $A_o = \frac{V_o}{2}$.

We may then write the complete solution as

$$V(r,\theta) = \frac{V_o}{2}\left[1 - \sum_{\ell=1}^{\infty}\frac{2\ell+1}{\ell}P_{\ell+1}(0)\left(\frac{r}{R}\right)^\ell P_\ell(\mu)\right].$$

To make a slightly different problem, assume that the boundary condition is specified as $V_o(\theta)$ on the surface at $r = R$. I will show the solution for inside only, but you can see how to extend the results to outside. The BC gives

$$V_o(\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell R^\ell\right) P_\ell(\cos\theta).$$

We can use the orthogonality condition again, but we cannot evaluate any integrals until we specify $V_o(\theta)$. Let's specify $V_o(\theta) = V_o \sin^2\theta = V_o(1 - \cos^2\theta)$. Why do you think we write it in terms of $\cos\theta$? When we apply the orthogonality condition, the entire equation simplifies to

$$A_\ell = \frac{2\ell + 1}{2R^\ell} \int_0^\pi V_o(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta.$$

We write

$$(1 - \cos^2\theta) = C_1 P_o(\cos\theta) + C_2 P_2(\cos\theta) = C_1 + C_2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right).$$

Then

$$(1 - \cos^2\theta) = C_1 - \frac{1}{2}C_2 + \frac{3}{2}C_2\cos^2\theta$$

and

$$C_1 - \frac{1}{2}C_2 = 1 \text{ with } -1 = \frac{3}{2}\ C_2.$$

Finally, $C_2 = -\frac{2}{3}$ and $C_1 = \frac{2}{3}$.

Formally, this calculation may also be done as an expansion of a function as a Legendre series given by

$$f(\theta) = \sum_{\ell=0}^{\infty} C_\ell P_\ell(\cos\theta).$$

It is an analog of the more familiar Fourier sine or cosine series. Why can we do this?

So far, we have dealt only with potentials that have some functional dependence on angle. How do we approach the problem if we have a surface charge distribution that is a function of angle? In equation form, suppose we are given $\sigma_o(\theta)$ on a spherical surface. This charge distribution is related directly to the electric field, not the potential. It gives rise to a discontinuity in the electric field, which, in turn, is related to the minus the gradient of the potential. Recall that the gradient in spherical coordinates is given by

$$\nabla V(r,\theta,\phi) = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}} = -\mathbf{E}(r,\theta,\phi).$$

Remember that the electric field produced by a surface charge density is given by $E = \frac{\sigma}{\epsilon_o}$. Maxwell's equations (ME) impose certain boundary conditions on the electric field according to

$$\oiint \mathbf{E}\cdot\mathbf{n}\, dA = \frac{q}{\epsilon_o} \Longrightarrow E_{2n} - E_{1n} = \frac{\sigma}{\epsilon_o} \; and \; \oint \mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt} \Longrightarrow E_{1t} = E_{2t}.$$

The potential must be continuous at the boundary, and the electric field discontinuity must give the surface charge density. Inside the sphere, $B_\ell = 0$ and outside the sphere, $A_\ell = 0$. Therefore, we may write the two BCs as

$$\sum_{\ell=0}^{\infty} A_\ell\, R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} B_\ell\, R^{-(\ell+1)} P_\ell(\cos\theta)$$

and

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)_{r=R} \Longrightarrow -\sum_{\ell=0}^{\infty}(\ell+1)B_\ell\, R^{-(\ell+2)} P_\ell(\cos\theta) - \sum_{\ell=0}^{\infty} \ell A_\ell\, R^{\ell-1} P_\ell(\cos\theta)$$
$$= -\frac{1}{\epsilon_o}\sigma_o(\theta).$$

We already know how A_ℓ and B_ℓ are related from the equation above so we obtain

$$\sum_{\ell=0}^{\infty}(2\ell+1)\, A_\ell R^{\ell-1} P_\ell(\cos\theta) = \frac{1}{\epsilon_o}\sigma_o(\theta)$$

We determine A_ℓ using the orthogonality of the Legendre polynomials as usual.

$$A_\ell = \frac{1}{2\epsilon_o R^{\ell-1}}\int_0^\pi \sigma_o(\theta) P_\ell(\cos\theta)\sin\theta\, d\theta$$

We need to know $\sigma_o(\theta)$ to proceed. When $\sigma_o(\theta)$ is a known function, we may expand it in a Legendre series to see which order $P_\ell(\cos\theta)$ contribute to the series. Then we see which terms survive to get the value for A_ℓ.

There is one more type of boundary value problem that does not get the attention that others get, but here is an example of this type.

Suppose you calculate the potential on the axis of a charged disk having radius R and constant surface charge density σ_o. The result is given by

$$V(r,0) = \frac{\sigma_0}{2\epsilon_o}\left(\sqrt{r^2+R^2} - r\right).$$

We would like to calculate the potential everywhere, even off the axis of symmetry. How to do that? We have already seen that the generating function for Legendre polynomials can be obtained from considering the expansion of a point charge located on the z-axis. This means that we can expand the potential for $r > R$ and match the Legendre polynomials on the z-axis to get the values for the coefficients. Here is how the process works for this case. Write

$$\sqrt{r^2+R^2} = r\left(1+\left(\frac{R}{r}\right)^2\right) = r\left[1+\left(\frac{1}{2}\right)\left(\frac{R}{r}\right)^2 - \left(\frac{1}{8}\right)\left(\frac{R}{r}\right)^4 + \cdots - 1\right]$$

We know that $V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{(\ell+1)}}\right)P_\ell(\cos\theta)$ and we also know that $A_\ell = 0$. Why? This means that

$$\sum_{\ell=0}^{\infty} B_\ell \frac{1}{r^{(\ell+1)}} = \frac{\sigma_0}{2\epsilon_o} r\left[1+\left(\frac{1}{2}\right)\left(\frac{R}{r}\right)^2 - \left(\frac{1}{8}\right)\left(\frac{R}{r}\right)^4 + \cdots - 1\right] = \frac{\sigma_0}{2\epsilon_o}\left(\frac{R^2}{2r} - \frac{R^4}{8r^3} + \cdots\right).$$

Use $P_\ell(0) = 1$ to obtain

$$B_o = \frac{\sigma_0 R^2}{4\epsilon_o}, B_1 = 0, and\ B_2 = -\frac{\sigma_0 R^2}{16\epsilon_o}, \cdots$$

Finally,

$$V(r,\theta) = \frac{\sigma_0 R^2}{4\epsilon_o}\left[\frac{1}{r} - \frac{R^2}{4r^3} P_2(\cos\theta) + \cdots\right].$$

NEXT TIME: Spherical harmonics, Bessel functions, and their use in BV problems.