Physics 6303 Lecture 10 September 19, 2018

LAST TIME: Sturm-Liouville problem and Laplace's equation in Cartesian coordinates

Laplace's equation in cylindrical coordinates

In cylindrical coordinates, the Laplacian operator is given by

$$\nabla^2 V(\rho,\phi,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

If we wish to solve a one-dimensional problem, notice that the operator is not symmetric in the variables, as it was in the Cartesian coordinate. This difference causes different results in one or two dimensions, depending on which dimension(s) we choose. Consider only ρ dependence.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V(\rho)}{\partial\rho}\right) = 0 \Longrightarrow \rho\frac{\partial V(\rho)}{\partial\rho} = A \text{ and } dV = A\frac{d\rho}{\rho}.$$

Therefore, $V(\rho) = A\ln\rho + B.$ Other one-dimensional cases are also possible. A cylindrical capacitor is an example of this behavior. Another interesting case is the two-dimensional case where z is not involved. The problem then is to solve

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0.$$

Using $V(\rho,\phi) = R(\rho)\Phi(\phi)$ as the form of the solution and simplifying, we obtain

$$\frac{1}{\rho}\Phi\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2}R\frac{d^2\Phi}{d\phi^2} = 0.$$

Divide by $R\Phi,$ simplify, and set each part to a constant to obtain

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = k^2 \text{ and } \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -k^2$$

What motivates our choice of signs for k^2?

Solving the $\Phi -$ equation, gives

$$\Phi(\phi) = A\cos k\phi \ + B\sin k\phi.$$

What values can k take on and why?

Why must k be an integer?

To solve the $R(\rho)$ equation try $R(\rho) = \rho^n$. Substitute our trial solution into the equation to obtain

$$\frac{d}{d\rho}(\rho n \rho^{n-1}) = n\rho \frac{d}{d\rho}\rho^n = n^2 \rho \rho^{n-1} = n^2 \rho^n = k^2 R.$$

Therefore, $n = \pm k$ and $R(\rho) = C\rho^k + D\rho^{-k}$ unless $k = 0$.

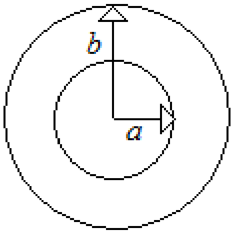
What is the problem when $k = 0$? How do we treat the $k = 0$ case? Use

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = 0 \Rightarrow \rho \frac{dR}{d\rho} = C_o \Rightarrow \frac{dR}{d\rho} = \frac{C_o}{\rho} \Rightarrow R = C_o \ln \rho + D_o.$$

Finally, we may write the complete solution as

$$V(\rho, \phi) = C_o \ln \rho + D_o + \sum_{m=1}^{\infty} (C_m \rho^m + D_m \rho^{-m})(A_m \cos m\phi \ + B_m \sin m\phi.)$$

These solutions are called cylindrical or zonal harmonics.



Example: Two infinitely long cylinders have inner radius a and outer radius b as shown. The inner cylinder is grounded so that the potential V is zero. The outer cylinder is held at a potential $V(\phi)$. Let ρ be the perpendicular distance from the cylinder axis to an arbitrary point. For parts (a) and (b), no detailed calculations are needed. (a) For $\rho < a$, write the basic structure of the solution. (b) For $\rho > b$, write the basic structure of the solution. (c) Calculate the potential for $a < \rho < b$. (d) Let $V(\phi) = V_o \sin \phi$ and calculate the potential for $a < \rho < b$.

Solution: For $\rho < a$, the solution at the center must remain finite. Therefore, C_o and D_m must be zero. The general form of the solution is given by

$$V(\rho, \phi) = D_o + \sum_{m=1}^{\infty} (C_m \rho^m)(A_m \cos m\phi \ + B_m \sin m\phi.)$$

For $\rho > b$, C_m and C_o must be zero to keep solution finite as $\rho \to \infty$. The general form of the solution is given by

$$V(\rho, \phi) = D_o + \sum_{m=1}^{\infty} (D_m \rho^{-m})(A_m \cos m\phi \ + B_m \sin m\phi.)$$

In between, $a < \rho < b$, there is no reason to set any of the constants to zero. Therefore,

$$V(a,\phi) = 0 = C_o \ln a + D_o + \sum_{m=1}^{\infty} (C_m a^m + D_m a^{-m})(A_m \cos m\phi\ + B_m \sin m\phi.)$$

$$V(b,\phi) = V(\phi) = C_o \ln b + D_o + \sum_{m=1}^{\infty} (C_m b^m + D_m b^{-m})(A_m \cos m\phi\ + B_m \sin m\phi.)$$

For the $m = 0$ terms, $0 = C_o \ln a + D_o$ and $V(\phi) = C_o \ln b + D_o$.

For $m \neq 0,$ we must apply the orthogonality conditions to isolate the remaining constants. Multiply by $\cos m\phi$ and $\sin m\phi$ and integrate to obtain

$$(C_m a^m + D_m a^{-m})A_m = 0 \text{ and } (C_m a^m + D_m a^{-m})B_m = 0,$$

$$(C_m b^m + D_m b^{-m})A_m = \frac{1}{\pi}\int_0^{2\pi} V(\phi) \cos m\,\phi\, d\phi$$

and

$$(C_m b^m + D_m b^{-m})A_m = \frac{1}{\pi}\int_0^{2\pi} V(\phi) \sin m\phi\, d\phi.$$

How would things look if $V(\phi) = \sin\phi$? What would you do if the potential were specified differently over a single conductor? How do you deal with the problem if the charge is specified instead of the potential? What is the relationship between the charge and the potential?

Before we attack the three-dimensional problem in cylindrical coordinates, I will look at spherical coordinates first to deal with a bit simpler problem. For the one-dimensional problem in spherical coordinates, things are not difficult. The general expression for the Laplacian in spherical coordinates is given by

$$\nabla^2 V(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right).$$

If $V(r,\theta,\phi) = V(r)$ only, then

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \Longrightarrow r^2 \frac{\partial V}{\partial r} = A \text{ and } V(r) = \frac{A}{r} + B.$$

This is just the spherically symmetric case that you dealt with even in your introductory course. The spherical capacitor is a good example of this case. We turn now to the case where

$$V(r,\theta,\phi) = V(r,\theta) \text{ with no } \phi - \text{dependence.}$$

These are the cases where we encounter a bit more difficulty in the differential equations. With no $\phi -$ dependence, the equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0.$$

As usual, we try to separate the variables and create two ODEs, one in r and one in θ. This process gives $V(r,\theta)=R(r)\Theta(\theta)$. Substituting this form into the equation above gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\big(R(r)\Theta(\theta)\big)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\big(R(r)\Theta(\theta)\big)}{\partial\theta}\right)=0.$$

Simplifying and dividing by $R(r)P(\theta)$ gives

$$\frac{1}{R(r)r^2}\frac{\partial}{\partial r}\left(r^2\frac{d\big(R(r)\big)}{dr}\right)+\frac{1}{\Theta(\theta)}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\big(\Theta(\theta)\big)}{d\theta}\right)=0.$$

Or, multiplying by r^2 gives

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{d\big(R(r)\big)}{dr}\right)+\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\big(\Theta(\theta)\big)}{d\theta}\right)=0.$$

Let

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{d\big(R(r)\big)}{dr}\right)=k \text{ and}$$

$$\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\big(\Theta(\theta)\big)}{d\theta}\right)=-k.$$

We do the indicated derivatives on the $R(r)$equation to get

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-kR=0.$$

Do you see what is interesting about this equation? It is sometimes called a Cauchy-Euler ODE.

Assume a solution of the form $R=r^m$ and substitute this value into the ODE to get

$$r^2m(m-1)r^{(m-2)}+2rmr^{(m-1)}-kr^m=0.$$

Factoring and simplifying gives

$$r^m(m^2+m-k)=0.$$

Therefore, $k=m(m+1)$, so if we substitute this back into the equation and resolve, we get, with $m=\ell$ and $m=-(\ell+1)$,

$$R(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{(\ell+1)}}.$$

Compare this particular substitution with the one you usually make for solving the damped harmonic oscillator problem, $x(t) = e^{rt}$. What does this substitution accomplish?

We may then rewrite the equation for $\Theta(\theta)$ as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\big(\Theta(\theta)\big)}{d\theta}\right) + \ell(\ell+1)\Theta(\theta) = 0,$$

where I have changed $m\ to\ \ell$ to make things look at bit more standard. We will attack the $\Theta(\theta)$ equation now. Let $\Theta(\theta) = P(\mu)$ with $\mu = \cos\theta$.
Use

$$d\mu = -\sin\theta\ \ d\theta$$

to get

$$-\frac{d}{d\mu}\left(-\sin^2\theta \frac{dP}{d\mu}\right) + \ell(\ell+1)P(\mu) = 0.$$

Finally,

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + \ell(\ell+1)P(\mu) = 0.$$

Notice that this is a Sturm-Liouville ODE with $f(\mu) = (1-\mu^2), g(\mu) = 0, w(\mu) = 1,$ and $\lambda = \ell(\ell+1)$. (By the way, the equations for the Cartesian coordinate problems and the cylindrical coordinate problems also have been Sturm-Liouville ODEs.) I will leave it to you to determine the values for $f, g, w,$ and λ.

So, what do we expect about the functions that we obtain from solving this equation? First, we notice that this equation has regular singular points at $\mu = \pm 1$. However, we should be able to find solutions if we expand about $\mu = 0$. We use the standard series solution assumption and write

$$P(\mu) = \sum_{n=0}^{\infty} a_n \mu^n.$$

It is best to expand the ODE by doing the indicated derivatives before substituting. This gives

$$\sum_{n=0}^{\infty} n(n-1)a_n \mu^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n \mu^n - 2\sum_{n=0}^{\infty} n a_n \mu^n + \ell(\ell+1)\sum_{n=0}^{\infty} a_n \mu^n = 0.$$

We now expand the series and set each of the coefficients of like powers of μ to zero. Here are some of the first few terms. I am setting $\ell(\ell+1) = k$ to make things more compact for now.

$ka_o - 2a_1\mu + ka_1\mu + 2a_2 - 2a_2\mu^2 - 4a_2\mu^2 + ka_2\mu^2 + 6a_3\mu - 6a_3\mu^3 + ka_3\mu^3 + 12a_4\mu^2$
-$12a_4\mu^4 - 8a_4\mu^4 + \cdots$
Only two terms are constant so $ka_o + 2a_2 = 0$. For the first power of $\mu, -2a_1 + ka_1 + 6a_3 = 0.$

There will be two arbitrary constants, a_o and a_1 because the ODE is second order. Each of these is related to an even and an odd coefficient, a_2 and $a_3,$ respectively.
We make everything more general by setting $n = p + 2$ in the first term and $n = p$ in the other terms. This change causes all of the coefficients of μ to be the same. We obtain

$$(p + 2)(p + 1)a_{p+2} - p(p - 1)a_p - 2pa_p + ka_p = 0.$$

We solve this equation for a_{p+2} in terms of a_p to obtain

$$a_{p+2} = \frac{p(p - 1) + 2p - k}{(p + 2)(p + 1)}a_p = \frac{p(p + 1) - \ell(\ell + 1)}{(p + 2)(p + 1)}a_p.$$

It is clear from here that this series terminates when $p = \ell.$ The series becomes a polynomial. However, it is useful to consider a slightly different way of writing this equation to see another feature. Expanding and refactoring the numerator gives

$$a_{p+2} = \frac{(p - \ell)(p + \ell + 1)}{(p + 2)(p + 1)}a_p.$$

We now know how to calculate everything in terms of a_o and a_1. Expressing the series out to μ^5 gives

$$P(\mu) = a_o\left[1 - \frac{\ell(\ell + 1)}{2!}\mu^2 + \frac{\ell(\ell + 1)(\ell - 2)(\ell + 3)}{4!}\mu^4 + \cdots\right]$$

$$+a_1\left[\mu - \frac{(\ell - 1)(\ell + 2)}{3!}\mu^3 + \frac{(\ell - 1)(\ell + 2)(\ell - 3)(\ell + 4)}{5!}\mu^5 + \cdots\right]$$

We have two linearly independent solutions with two arbitrary constants, so you might think we are finished, but what is the problem? For any integer value of $\ell,$ one of the series diverges at $x = \pm 1,$ whereas the other one terminates as a polynomial. What is the advantage of having a polunomial? (Hint: Think of analyticity. What types of functions are analytic everywhere?) For $\ell = 0,$ the a_1 series gives

$$Q_o = a_1\left(\mu + \frac{\mu^3}{3} + \frac{\mu^5}{5} + \cdots\right) = \frac{a_1}{2}\ln\left(\frac{1 + \mu}{1 - \mu}\right).$$

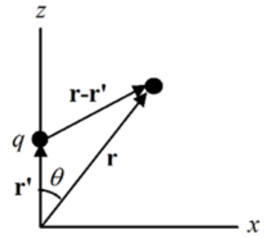
Similar analyses for other values of ℓ lead to the Legendre functions of the second kind $Q_\ell(\mu)$, which converge only for the interval $-1 < x < 1$ and are not used in solving potential problems with Laplace's equation. Finally, we arrive at the solution for azimuthal symmetric problems to be

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell \frac{1}{r^{(\ell+1)}} \right) P_\ell(\mu).$$

The beauty of the Sturm-Liouville problem is that we already know we have a complete set of orthogonal functions so that

$$\int_{-1}^{1} P_\ell(\mu) P_m(\mu) d\mu = 0 \ \text{ for } \ell \neq m.$$

What we do not know, however, is the value when $\ell = m$. This expression is best obtained by using the generating function, which is, as I mentioned before, nothing more than the expansion of the potential of a point charge displaced from the origin. Write the potential at point **r** for a point charge located at $\mathbf{r}'$ to obtain



$$V(r) = \frac{q}{4\pi\epsilon_o} \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{q}{4\pi\epsilon_o} \frac{1}{(r^2 - 2\mathbf{r} \cdot \mathbf{r}' + \mathbf{r}'^{\mathbf{2}})^{1/2}}.$$

Let $\mathbf{r}' = s\hat{\mathbf{z}}$ and expand the denominator for $r > s$ to obtain (homework problem)

$$\frac{1}{\sqrt{1 - 2x\mu + x^2}} = \sum_{\ell=0}^{\infty} A_\ell P_\ell(\mu),$$

where $x = \frac{r}{s}$ and $A_\ell = \frac{q}{4\pi\epsilon_o s^{\ell+1}}$ with $B_\ell = 0$. The generating function $G(x,\mu)$ is then defined as

$$G(x,\mu) = \frac{1}{\sqrt{1 - 2x\mu + x^2}}.$$

The generating function is one method to use for determining $\int_{-1}^{1} P_\ell(\mu) P_m(\mu) d\mu$ when $\ell = m$.

The result is given by

$$\int_{-1}^{1} P_\ell(\mu) P_\ell(\mu) d\mu = \frac{2}{2\ell + 1} \ .$$

That means we may write the orthogonality condition in general as

$$\int_{-1}^{1} P_\ell(\mu) P_m(\mu) d\mu = \frac{2}{2\ell + 1} \delta_{\ell m} \ .$$

It is very useful to find some relationships between the various different Legendre polynomials. These relationships are called recursion relations and, as I will demonstrate with an example are quite useful at times. Once again, the generating function may be used to generate these relationships. I will list a few of these at the beginning of the next lecture when we do some example problems.

NEXT TIME: Recursion relationships, examples, and spherical harmonics