LAST TIME: EM theory of light, Maxwell's equations, wave equations for **E** and **B**, $\frac{E_o}{B_o} = c$

$\boldsymbol{E} \perp \boldsymbol{B} \perp \boldsymbol{k}$ and $\boldsymbol{S}$, $\langle e^{i\omega t}\rangle_T = e^{i\omega t}\text{sinc}\left(\frac{\omega T}{2}\right)$, $\langle \cos^2 \omega t\rangle_T = \frac{1}{2}[1 + \text{sinc}\,\omega T \cos 2\omega t]$

$\mathbf{S} = \frac{1}{\mu_o}\mathbf{E} \times \mathbf{B} = \mathrm{c}^2\epsilon_o\mathbf{E} \times \mathbf{B}$

In optics, we usually refer to the time-averaged Poynting vector as the irradiance. It is the same as intensity when general EM waves are considered. Therefore,

$$\langle|\mathbf{S}|\rangle_T = c^2\epsilon_o|\mathbf{E}_o \times \mathbf{B}_o|\langle\cos^2 \mathbf{k}\cdot\mathbf{r} - \omega t\rangle_T = \frac{c^2\epsilon_o}{2}|\mathbf{E}_o \times \mathbf{B}_o| \text{ for } \omega T \gg 1.$$

Finally,

$$I = \frac{c^2\epsilon_o}{2}\frac{E_o^2}{c} = \frac{c\epsilon_o}{2}E_o^2.$$

What is the expression for the electric field of a HeNe laser operating at 1 mW? Calculate B as well. We will use a beacm diameter of 1 mm with a wavelength of 633 nm. We write the electric field as

$$\mathbf{E} = \mathbf{E}_o e^{i(kz-\omega t)}.$$

$$\langle S\rangle = \frac{(1\times 10^{-3})4}{\pi(1\times 10^{-3})^2} = \frac{c\epsilon_o}{2}E_o^2 = 1.3\times 10^{-3}\,\frac{W}{m^2}$$

$$E_o = \sqrt{\left[\frac{2\langle S\rangle}{c\epsilon_o}\right]} = \left[\frac{2(1.3\times 10^{-3})}{(3\times 10^8)(8.85\times 10^{-12})}\right]^{1/2} = 980\,\frac{\mathrm{V}}{\mathrm{m}}.$$

Then

$$B_o = \frac{E_o}{c} = 3.3\times 10^{-6}\ \mathrm{T}$$

$$k = \frac{2\pi}{\lambda} = 9.9\times 10^6\ \mathrm{m}^{-1} \text{ and } \omega = 3\times 10^{15}\ \mathrm{rad/s}$$

Therefore, our wave is

$$\mathbf{E} = 980\,\hat{\mathbf{x}}\,e^{i\left(9.9\times 10^6 z - 3\times 10^{15} t\right)}\ \mathrm{V/m}$$

and

$$\mathbf{B} = 3.3\times 10^{-6}\,\hat{\mathbf{y}}\,e^{i\left(9.9\times 10^6 z - 3\times 10^{15} t\right)}\ \mathrm{T}$$

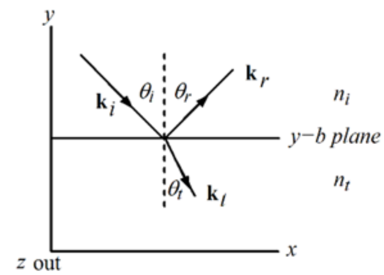
In a material such as glass, the irradiance takes on a slightly different to account for being in a material instead of a vacuum. The expression becomes

$$I = \epsilon v\langle E^2\rangle_T$$

Recall that $v = \frac{c}{n} = \lambda_m \nu_m$, but $\nu_m = \nu_{vac}$ and $\frac{c}{\lambda_{vac}} = \frac{c}{n\lambda_m}$. Therefore, $\lambda_m = \frac{\lambda_{vac}}{n}$.

We are now in a position to understand why the amplitude in a spherical wave must vary as (1/*r*). Because the energy is conserved in the wave, and the area of a sphere increases as r^2, the product of the surface area and the irradiance must be constant. Similarly, for the cylindrical wave, the surface area varies as *r*, so the amplitude of the wave must vary as $\frac{1}{\sqrt{r}}$.

Now we want to look at the way light interacts with boundaries. We know that Snell's law and the law of reflection predict where light goes when it encounters a boundary, but neither of them tell us how much goes where. We are now in a position to determine the answer to that question. The figure shows the k vectors for the incident wave, the refpected wave, and the transmitted wave. Notice that we have not yet put directions of **E** or **B** in the figure. The only requirement is that the three vectors form a right-handed Cartesian coordinate system. Let's write each of the incident electric fields as expressions given by



$$\mathbf{E}_i = \mathbf{E}_{oi}e^{i(\mathbf{k}_i\cdot\mathbf{r}-\omega_i t)}, \qquad \mathbf{E}_r = \mathbf{E}_{or}e^{i(\mathbf{k}_r\cdot\mathbf{r}-\omega_r t+\delta_r)}, \text{and } \mathbf{E}_t = \mathbf{E}_{ot}e^{i(\mathbf{k}_t\cdot\mathbf{r}-\omega_t t+\delta_t)}.$$

So long as **E** is perpendicular to **B** and **k**, we are OK. It turns out that Maxwell's equations make some interesting conditions on E and B at an interface. They are most easily used in integral form to derive what are called boundary conditions. Here are the boundary that apply when an EM wave crosses a dielectric boundary.

1. The tangential component of the electric field **E** in continuous. Being continuous means that the sum of the fields ono one side of the boundary must equal the sum of the fields on the other side of the boundary. This is usually written as $E_{1t} = E_{2t}$. It follows from Faraday's law as the area of a small rectangle vanishes when the line integral shrinks.

2. The normal (perpendicular) component of the magnetic field B is continuous across the boundary. This is usually written as $B_{1n} = B_{2n}$ and follows from Gauss's law for magnetic fields.

3. The normal component of ε**E** is continuous across the boundary, where ε is the permittivity of the material. ε is related to the index of refraction, so we will convert our final results to *n*. This is written in equation form as $\epsilon_1 E_{1n} = \epsilon_2 E_{2n}$.

4. The tangential component of $\frac{\mathbf{B}}{\mu}$ is continuous across the boundary, where μ is the permeability of the material. μ is usually set to μ_o because most optical materials do not have strong magnetic properties. The equation form of this boundary condition is given by $\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$.

It turns out that the existence of BCs provides us with information. The BCs must hold at all times and at all locations along the boundary. Therefore, we see that $\omega_i = \omega_t = \omega_r$. That the frequencies must be the same in classical physics also arises from the fact that the frequency is determined by a counting process – so many oscillations per second. The spatial part yields
$\mathbf{k}_i \cdot \mathbf{r} = (\mathbf{k}_r \cdot \mathbf{r} + \delta_r) = (\mathbf{k}_t \cdot \mathbf{r} + \delta_t)$ evaluated at the boundary $y = b$.

Therefore, $(\mathbf{k}_i - \mathbf{k}_r) \cdot \mathbf{r} = \delta_r$ and $(\mathbf{k}_i - \mathbf{k}_t) \cdot \mathbf{r} = \delta_t$. r sweeps out a plane (the surface) perpendicular to $(\mathbf{k}_i - \mathbf{k}_r)$ so $(\mathbf{k}_i - \mathbf{k}_r) \parallel \hat{\mathbf{u}}_n$, where $\hat{\mathbf{u}}_n$ is the unit normal to the surface. Finally, $(\mathbf{k}_i - \mathbf{k}_r) \times \hat{\mathbf{u}}_n = 0$ and $k_i \sin\theta_i = k_r \sin\theta_r$, but $k_i = k_r$, so we obtain the law of reflection

$$\theta_i = \theta_r.$$

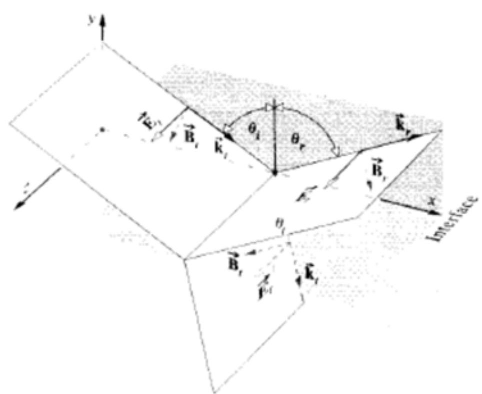
By a similar argument, we also obtain

$$k_i \sin\theta_i = k_t \sin\theta_t.$$

Now, however, the ks donot have the same magnitude, but we can multiply both sides by c/ω to get

$$n_i \sin\theta_i = n_t \sin\theta_t.$$

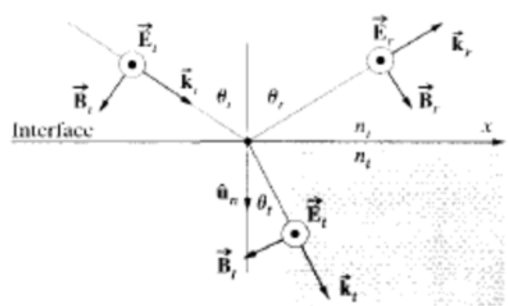
Because of the vector nature of our approach, we also see that all of the k vectors and the unit normal to the plane lie in a plane that we call the plane of incidence. To make use of the nature of the BCs, we need to decide on a direction of the electric field. The following figures from Hecht show one choice of the direction of **E**. This choice is referred to as **E** perpendicular to the plane of incidence. We do need to remember, however, that when we are in a material, $E = vB$, instead of cB. For our particular choice of the field directions, our BCs become



$$E_{oi} + E_{or} = E_{ot}$$

and

$$-\frac{B_{oi}}{\mu_i}\cos\theta_i + \frac{B_{or}}{\mu_i}\cos\theta_r = -\frac{B_{ot}}{\mu_t}\cos\theta_t.$$



The second BC comes from the continuity of the tangential component of $\mathbf{B}/\mu$. As I mentioned earlier, most optical materials are nonmagnetic so $\mu = \mu_o$. As our last step, we convert B to E using $E = vB$. We now have two equations is two unknowns that we may solve for the ratios of the reflected E field to the incident E field and the ratio of the transmitted E field to the incident E field. These are known as the Fresnel coefficients (equations) because Fresnel derived them before Maxwell's equations were known. Their validity extends to linear, homogeneous, isotropic media. Here they are:

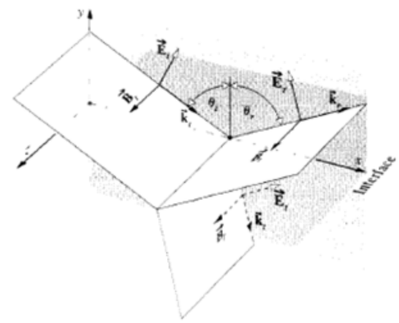
$$r_\perp = \left(\frac{E_{or}}{E_{oi}}\right)_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

and

$$t_\perp = \left(\frac{E_{ot}}{E_{oi}}\right)_\perp = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$r_\perp$ is the amplitude reflection coefficient and $t_\perp$ is the amplitude transmission coefficient.

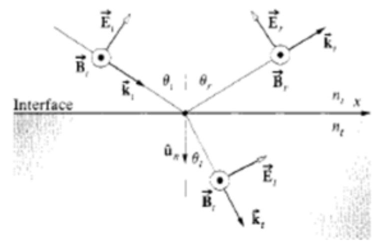
The electric field may lie in the plane of incidence or be parallel to it. Here are the figures from Hecht that show this case. For this case, the magnetic fields have all been selected to have the same direction, consistent with **E** and **k**. This time, we apply the same BCs, but the equations are different because the electric has been rotated by 90 degrees.



$$E_{oi}\cos\theta_i - E_{or}\cos\theta_r = E_{ot}\cos\theta_t$$

and

$$\frac{1}{\mu_i v_i}E_{oi} + \frac{1}{\mu_r v_r}E_{or} = \frac{1}{\mu_t v_t}E_{ot}$$



Again, we assume that the material is nonmagnetic so we finally obtain

$$r_{\parallel} = \left(\frac{E_{or}}{E_{oi}}\right)_{\parallel} = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i}$$

and

$$t_{\parallel} = \left(\frac{E_{ot}}{E_{oi}}\right)_{\parallel} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}.$$

Sometimes, we would like to write these entirely in terms of the angles rather than involve the indices of refraction. We use Snell's law to do this and obtain

$$r_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \quad r_{\parallel} = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

and

$$t_{\perp} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i + \theta_t)} \quad t_{\parallel} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}.$$

Let's look at the case for normal incidence where $\theta_i = \theta_r = \theta_t = 0$. We note that

$$r_{\perp} = \frac{n_i - n_t}{n_i + n_t} = -r_{\parallel}.$$

The reason that these do not result in the same value is that the notion of the plane of incidence is no longer applicable because everything has collapsed to a line. On the other hand, we do find that, at $\theta_i = \theta_r = \theta_t = 0$

$$t_{\perp} = t_{\parallel} = \frac{2n_i}{n_i + n_t}.$$

Let's point out one thing now that is a bit odd, but we will come back to it later and explain what is going on. If we calculate the reflection and transmission coefficients for light incident from air to glass we find that

$$r_\perp = \frac{1-1.5}{1+1.5} = -0.2 \text{ and } t_\perp = \frac{2(1)}{2.5} = 0.8,$$

but when we go from a high index medium to a low index medium, we find that

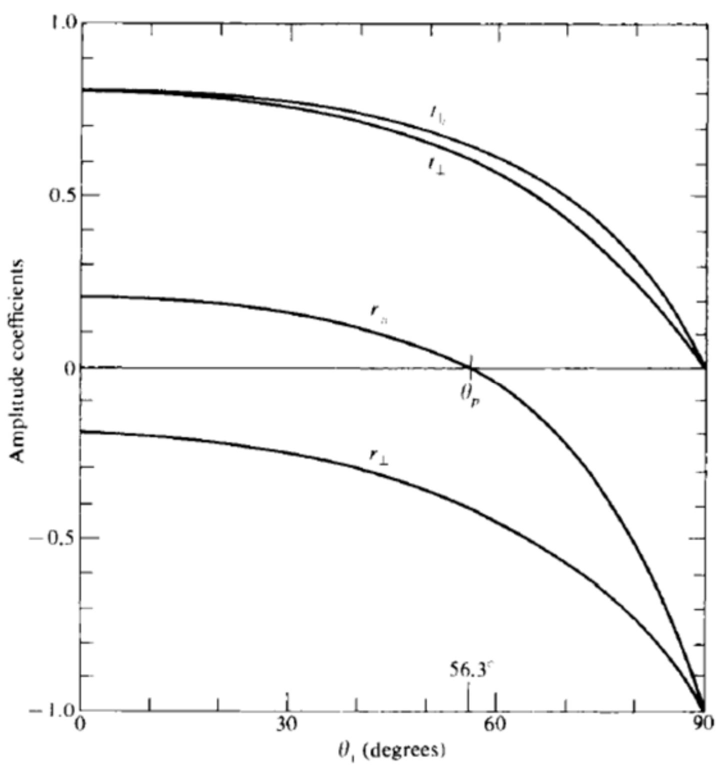
$$r_\perp = \frac{n_i - n_t}{n_i + n_t} = \frac{1.5-1}{1.5+1} = +0.2 \text{ and } t_\perp = \frac{2(1.5)}{2.5} = 1.2.$$

This seems odd to have a transmission amplitude coefficient great than 1, but we will see when we calculate the irradiance, all is well. Its occurs because the energy flow rate is slower in the higher index medium than in air. The negative sign in the reflection coefficient means that a phase change has occurred, and the reflected electric field is reversed in direction compared to the incident field.

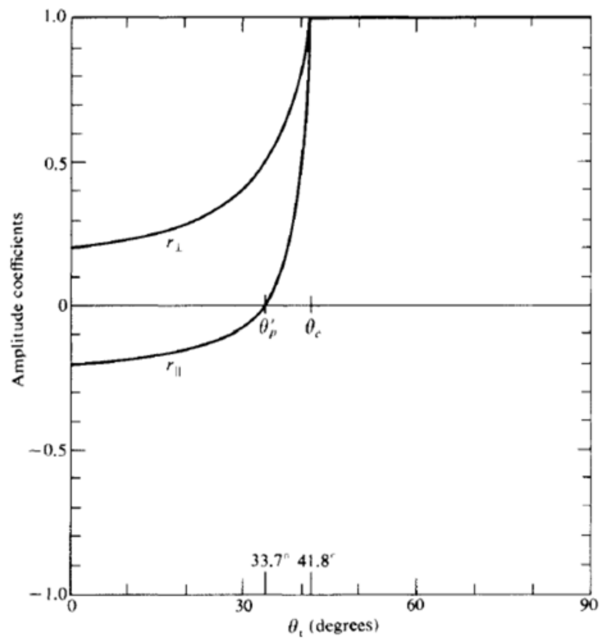
Because Snell's law follows from the electromagnetic treatment, there are no surprises concerning the critical angle for total internal reflection, but there is one thing that does not show up in the ray treatment. Consider the expression given by

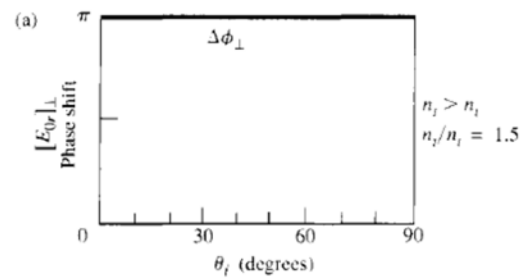
$$r_\parallel = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}.$$

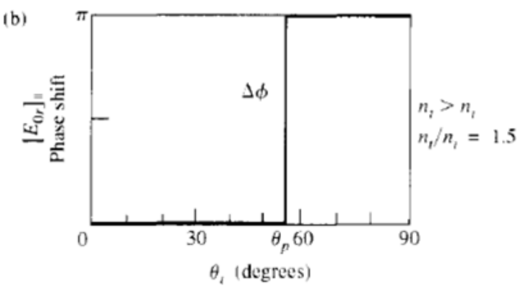
When $\theta_i + \theta_t = 90^o, r_\parallel = 0$. The value of θ_i for which this occurs is called the polarizing angle or Brewster angle because only light polarized perpendicular to the plane of incidence is reflected. Let's look at some graphs from Hecht to see how the angular dependence of the coefficients behaves. The cases shown in the figure to the right are for external reflections ($n_t > n_i$), where we go from air to glass. There are some key things to notice here. $r_\perp$ is always negative, but $r_\parallel$ changes sign at the polarizing angle. The transmission coefficients are always positive. The next page has the coefficients for the case on an internal reflection where $n_t < n_i$.

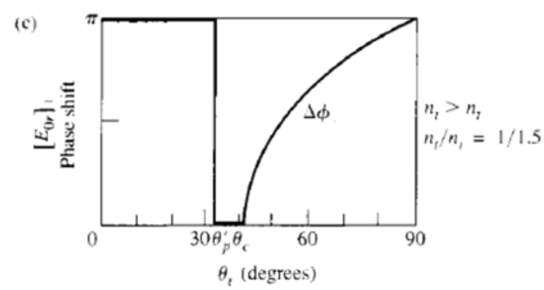


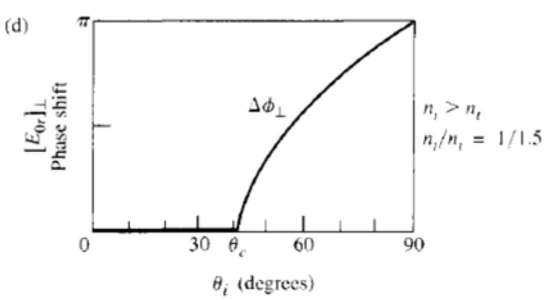
These are the reflection coefficients only because the transmission coefficients do not show any particularly interesting behavior. Note that for an internal reflection, $r_\perp$ is always positive, but the parallel coefficient changes from negative to positive at the polarizing angle.

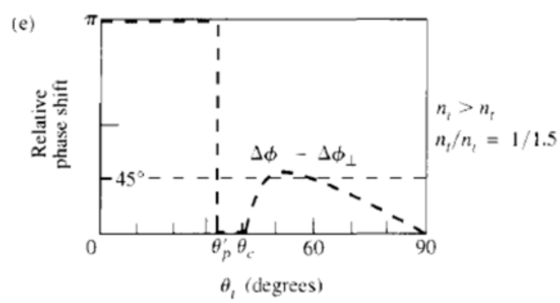












NEXT TIME: Examination 1covering geometrical optics.