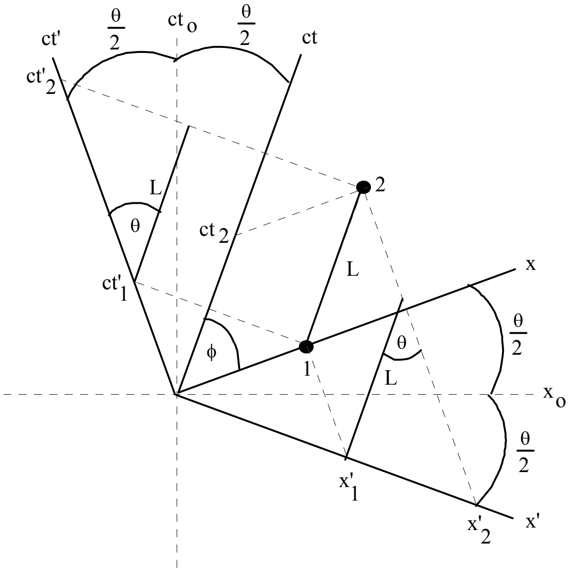
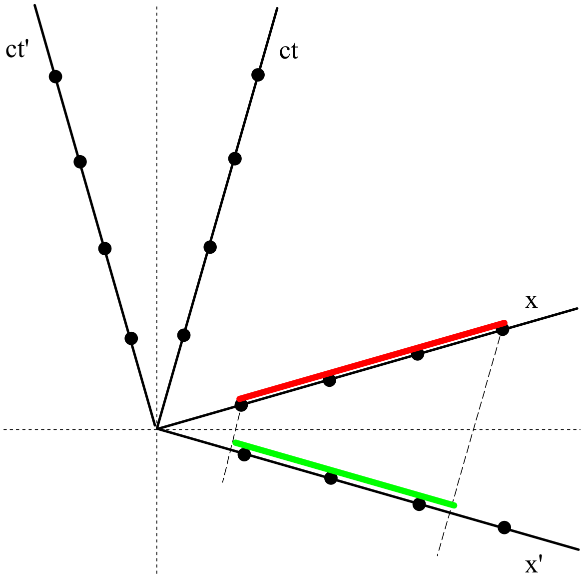
Physics 6303 Lecture 9 September 17, 2018

LAST TIME: Finished tensors, vectors, 4-vectors, and 4-tensors

One last point is worth mentioning although it is not commonly in use. It does, however, build on the idea of contravariant and covariant vectors so I will comment briefly on it. You might recall that we usually say that no frame of reference is any better and any other inertial frame of reference. But, on the other hand, we construct one coordinate system as orthogonal and the other as oblique. This always bothered me when I was a student. It turns out that it is possible to represent both coordinate systems as oblique coordinate systems and use the ideas of contravariant vector components to study the geometry of special relativity. Such diagrams are called Loedel diagrams, and they look like the figures to the right. Now consider how length contraction is represented using the Loedel diagram. Figure 6 shows the length contraction, but because unit length is the same in both S and S′, the effect may be calculated or measured with a ruler without any scaling factor.





Length contraction from ct - x to ct' - x'

At the beginning of the course, I wrote several partial differential equations (PDEs) that are used in many fields of physics. Now, we are going to determine how to deal with these equations. As you know, partial differential equations are much more difficult to solve that ODEs because when you undo the partial derivative, you do not get a constant, you obtain an arbitrary function of the other variables. For example, consider the very simple PDE given by

$$\frac{\partial f(x,y,z)}{\partial x} = 0.$$

The most general solution to this equation is $f(x,y,z) = g(y,z) + C.$ Without additional information, there is very little we can do to solve this equation. We will do a bit more with this later after we complete some more traditional problems. We start with Laplace's equation because it has many applications and you have probably already dealt with it in some of your undergraduate courses, especially EM theory. Before we start with the specifics of solutions to the various PDEs, I want to show you a very general problem that will occur in many of the specific situations. It is called the Sturm-Liouville problem. You will see this general theme over and over in our solutions to many of the equations of physics. Recall that most of the solutions to the PDEs of physics are solved by the separation of variables technique. This technique reduces the PDE to a set of ODEs, depending on the dimensionality of the problem. Usually, at least one of the ODEs has the following form given by

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] - g(x)\,y + \lambda\,w(x)\,y = 0.$$

This is a Sturm-Liouville ODE that is linear and second order. The boundary conditions that go with the equation are given by

$$\alpha_1 y + \beta_1\frac{dy}{dx} = 0 \text{ at } a \text{ and } \alpha_2 y + \beta_2\frac{dy}{dx} = 0 \text{ at } b.$$

$w(x) \geq 0\ over\ a \leq x \leq b$. α and β cannot both be zero.

If $\alpha = 0$, we have Neumann BC, and if $\beta = 0$, we have Dirichlet BC. One of the most important properties of this equation is the orthogonality of the eigenfunctions. To see this consider the equation for two solutions y_m and y_n given by

$$\frac{d}{dx}\left[f(x)\frac{dy_m}{dx}\right] - g(x)\,y_m + \lambda_m\,w(x)\,y_m = 0$$

and

$$\frac{d}{dx}\left[f(x)\frac{dy_n}{dx}\right] - g(x)\,y_n + \lambda_n\,w(x)\,y_n = 0.$$

Multiply the top equation by y_n, the bottom equation by y_m, and subtract to get

$$y_n\frac{d}{dx}\left[f(x)\frac{dy_m}{dx}\right] - y_m\frac{d}{dx}\left[f(x)\frac{dy_n}{dx}\right] + (\lambda_m - \lambda_n)w(x)y_m y_n = 0.$$

Integrate over the range $[a, b\ ]$ to get

$$\int_a^b \left\{y_n\frac{d}{dx}\left[f(x)\frac{dy_m}{dx}\right] - y_m\frac{d}{dx}\left[f(x)\frac{dy_n}{dx}\right]\right\}dx = (\lambda_n - \lambda_m)\int_a^b w(x)y_m y_n\,dx$$

Next, integrate the LHS by parts to get

$$\left\{y_n\left[f(x)\frac{dy_m}{dx}\right]\right\}\Big|_a^b - \int_a^b f(x)\frac{dy_n}{dx}\frac{dy_m}{dx}dx - \left\{y_m\left[f(x)\frac{dy_n}{dx}\right]\right\}\Big|_a^b + \int_a^b f(x)\frac{dy_m}{dx}\frac{dy_n}{dx}dx$$

Terms 1 and 3 vanish for either Dirichlet or Neumann BCs, and terms two and four cancel because they are symmetric in *m* and *n*. For mixed BCs, we have to be sure that the first and third terms cancel. Use the BCs and substitute to get

$$f(b)y_n(b)\left[-\frac{\alpha_2}{\beta_2}y_m(b)\right]-f(a)y_n(a)\left[-\frac{\alpha_1}{\beta_1}y_m(a)\right]-f(b)y_m(b)\left[-\frac{\alpha_2}{\beta_2}y_n(b)\right]$$
$$+f(a)y_m(a)\left[-\frac{\alpha_1}{\beta_1}y_n(a)\right]=0=(\lambda_n-\lambda_m)\int_a^b w(x)y_m y_n\,dx.$$

So long as $\lambda_n \neq \lambda_m, \int_a^b w(x)y_m(x)y_n(x)\,dx = 0.$ $y_m(x)$ *and* $y_n(x)$ are orthogonal with respect to the weighting function $w(x)$. Two other important cases exist where the LHS is zero. If $f(b) = f(a) = 0,$ then the LHS is zero and the BCs do not matter as long as $y(x)$ is finite. Finally, if the product $f(x)y(x)\frac{dy(x)}{dx}$ is periodic $b - a,$ then the LHS is zero. This means we can expand any function as

$$f(x)=\sum_{n=0}^{\infty} a_n\, y_n(x), \text{where } a_n \text{ is given by}$$

$$a_m=\frac{\int_a^b f(x)y_m(x)w(x)dx}{\int_a^b [y_m(x)]^2\, w(x)dx}.$$

Substitute the value for a_m back into the expression for $f(x)$ to get

$$f(x)=\sum_{n=0}^{\infty}\frac{\int_a^b f(x')y_n(x')w(x')dx'}{\int_a^b [y_n(x)]^2\, w(x)dx}y_n(x).$$

Interchange the sum and the integral to obtain

$$f(x)=\int_a^b f(x')\sum_{n=0}^{\infty}\frac{y_n(x)y_n(x')w(x')dx'}{\int_a^b [y_n(x)]^2\, w(x)dx}.$$

For the RHS to give

$$f(x),\qquad \sum_{n=0}^{\infty}\frac{y_n(x)y_n(x')w(x')dx'}{\int_a^b [y_n(x)]^2\, w(x)dx}=\delta(x-x'), \text{the completeness relation.}$$

I will ask you to prove that the eigenvalues are real as a homework assignment.

Summary of what all this means:

Any differential equation that has this form is an eigenvalue problem. The eigenfunctions form a complete set of orthogonal functions that are usually normalized. The solution to the problem can be expressed as a superposition of these eigenfunctions whose arguments are determined by the eigenvalues. The orthogonality of the eigenfunctions guarantees that we can find the values for the $a_n's.$

Laplace's equation in Cartesian coordinates

$$\nabla^2 V(x,y,z) = 0,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Our partial differential equation to solve is then

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

If $V(x,y,z)$ is a function of one variable only, x, y, or z, then the solution is easy. The PDE directly becomes an ODE of the form

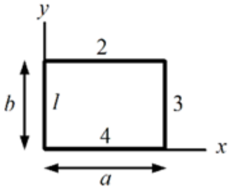
$$\frac{d^2 V(z)}{dz^2} = 0 \text{ and } V(z) = Az + B \text{ with } A \text{ and } B \text{ determined by BCs.}$$

If $V(x,y,z)$ is a function of only two of the variables, say x and y, then

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

The classic situation for this form of Laplace's equation is a long rectangular tube with open ends so that z never enters into the problem. Comment on symmetry. We need BCs to obtain the constants involved in the solution of the ODEs after we apply the usual separation of variables technique. How do we know that the separation of variables technique will work?
Let's label the potential on each of the four surfaces, 1, 2, 3, and 4 as

$$V_1, V_2, V_3, \text{and } V_4.$$



You will notice that these BCs are not homogeneous, so our Sturm-Liouville problem does not seem to work here. We certainly expect that we can separate the variables, but how do we deal with this problem of nonhomogeneous BCs? Hint: We invoke superposition. If $V_1, V_2,$ and $V_3 = 0,$ we then have homogeneous BCs, and we can solve for $V_4(x,y)$ using Laplace's equation. Then, we solve for $V_3(x,y)$ with $V_1, V_2,$ and $V_4 = 0.$ As you might expect, we then get $V_2(x,y)$ and $V_1(x,y)$ with the other potentials set to zero. The final solution is the superposition of $V_1(x,y), V_2(x,y), V_3(x,y),$ and $V_4(x,y).$ The beauty of linear ODEs is that they allow for linear superposition of solutions. What is another important property of linear equations? We use separation of variables to write

$$V(x,y) = X(x)Y(y) \text{ with each function having only one variable.}$$

This means that partial derivatives become derivatives to get

$$\frac{\nabla^2 V_4(x,y)}{V_4(x,y)} = \frac{\frac{\partial^2 V_4(x,y)}{\partial x^2} + \frac{\partial^2 V_4(x,y)}{\partial y^2}}{V_4(x,y)} = \frac{Y(y)\frac{d^2X(x)}{dx^2}}{X(x)Y(y)} + \frac{X(x)\frac{d^2Y(y)}{dy^2}}{X(x)Y(y)} = 0$$

Simplifying gives

$$\frac{\frac{d^2X(x)}{dx^2}}{X(x)} + \frac{\frac{d^2Y(y)}{dy^2}}{Y(y)} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0 \Rightarrow \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}.$$

This condition can be true only if

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k^2.$$

What about the choice of the sign of k^2? These equations are certainly both Sturm-Liouville ODEs, and the $X(x)$ equation has homogeneous BCs. The $X(x)$ equation has solutions of sines and cosines, whereas the $Y(y)$ equation has solutions of hyperbolic sines and cosines or exponential functions. They are

$$X(x) = A\cos kx + B\sin kx \text{ and } Y(y) = C\cosh ky + D\sinh ky.$$

Our BCs are $V_4(x=0,y)=0, V_4(x=a,y)=0, V_4(x,y=b)=0,$ and $V_4(x,y=0)=V_4(x)$.

Therefore, $V_4(x,y) = (A\cos kx + B\sin kx)(C\cosh ky + D\sinh ky)$. We need to apply the BCs to get a nontrivial solution. Neither *A* = *B* = 0 nor *C* = *D* = 0 can occur.

$$V_4(0,y) = (A\cos k0 + B\sin k0)(C\cosh ky + D\sinh ky) = 0.$$
$$= (A)(C\cosh ky + D\sinh ky) = 0.$$

We cannot have *C* = *D* = 0, so we choose *A* = 0. Apply the next BC to get

$$V_4(a,y) = (B\sin ka)(C\cosh ky + D\sinh ky) = 0$$

We cannot choose *B* = 0 or *C* = *D* = 0 or we get a trivial solution. We must choose $\sin ka = 0$, which means that $ka = n\pi$. This gives our eigenvalues $k = \frac{n\pi}{a}$. Along side 2, we get

$$V_4(x,b) = \left(B\sin\frac{n\pi}{a}x\right)\left(C\cosh\frac{n\pi}{a}b + D\sinh\frac{n\pi}{a}b\right) = 0,$$

so

$$C = -D\tanh\frac{n\pi}{a}b.$$

The last BC condition gives $BC\sin\frac{n\pi}{a}x = V_4(x,0) = V_4(x)$. No single function can possibly satisfy this condition, so we need to use a superposition to satisfy it. This essentially amount to expressing the last BC as a Fourier series given by

$$V_4(x,0) = V_4(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{a}x.$$

Because we have reduced this problem to a Sturm-Liouville system, we know the eigenfunctions are orthogonal with an orthogonality condition given by

$$\int_0^a \sin\frac{n\pi}{a}x \, \sin\frac{m\pi}{a}x dx = \frac{a}{2}\delta_{mn}.$$

We get for *Cn*

$$C_n = \frac{2}{a}\int_0^a V_4(x) \sin\frac{n\pi}{a}x dx \text{ so}$$

the complete solution for this part of the problem is given by

$$V_4(x,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{a}x \left(\cosh\frac{n\pi}{a}y - \coth\frac{n\pi}{a}b \sinh\frac{n\pi}{a}y\right).$$

To complete the problem, we must repeat this process three more times to get the full solution for arbitrary $V_1, V_2,$ and V_3. We cannot compute *Cn* until we know the exact form of each potential on the boundary. Here are the important ideas to take away from this example. First, even if the BCs are not homogeneous, you can make them so and use superposition. This means that no matter what these BCs are, you can solve this type of problem. Admittedly, it is tedious, but the process is clear.

Extension to three dimensions in Cartesian coordinates is relatively straightforward, but for completeness, here is how things work.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Once again, let $V(x,y) = X(x)Y(y)Z(z)$. Separation of variables gives

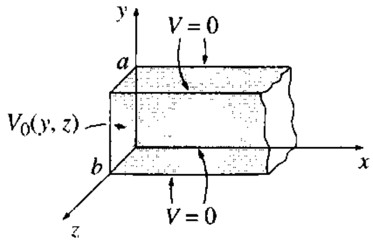
$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0 \text{ with}$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k_x^2; \quad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = k_y^2; \quad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = k_z^2$$

and

$$k_x^2 + k_y^2 + k_z^2 = 0.$$

To proceed, we need a specific problem to solve, so consider the figure shown with the boundary conditions given. Because some BCs are zero, this already fits the model of a Sturm-Liouville problem. Here are our BCs; $V = 0$ when $y = 0, y = a, z = 0,$ and $z = b. V \to 0$ when $x \to \infty$ and $V = V_o(y, z)$ when $x = 0.$ Because $.V \to 0$ when $x \to \infty,$ we need an exponential function for the x – variable. With homogeneous BCs in the y and z directions, we expect sines and cosines as the solutions. Therefore, we may write



$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k_y^2; \quad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -k_z^2; and\ \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k_x^2 = k_y^2 + k_z^2.$$

We get $X(x) = Ae^{\sqrt{k_y^2+k_z^2}x} + Be^{-\sqrt{k_y^2+k_z^2}x}$, $Y(y) = C\sin k_y y + D\cos k_y y$, and

$$Z(z) = E\sin k_z z + F\cos k_z z.$$

$$V \to 0 \text{ when } x \to \infty \implies A = 0, V = 0 \text{ when } y = 0 \implies D = 0, V = 0 \text{ when } z = 0 \implies F = 0.$$

$$V = 0 \text{ when } y = a \implies k_y = \frac{n\pi}{a} \text{ and } V = 0 \text{ when } z = b \implies k_z = \frac{m\pi}{b}.$$

Combining the remaining constant into C gives

$$V(x, y, z) = Ce^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}x}\sin\frac{n\pi}{a}y\sin\frac{m\pi}{b}z.$$

We are left with determining *C* so we write

$$V(0, y, z) = \sum_{n,m}^{\infty} C_{n,m}\sin\frac{n\pi}{a}y\sin\frac{m\pi}{b}z = V_o(y, z).$$

Because sines are orthogonal eigenfunctions, we may calculate $C_{n,m}$.
We use the standard method of multiplying by another eigenfunction and using the orthogonality condition to obtain

$$\sum_{n,m}^{\infty} C_{n,m}\int_0^a \sin\frac{n\pi}{a}y\sin\frac{n'\pi}{a}y\,dy\int_0^b \sin\frac{m\pi}{b}z\sin\frac{m'\pi}{b}z\,dz$$

$$= \int_0^a\int_0^b \mathrm{V}_o(\mathrm{y},\mathrm{z})\sin\frac{m'\pi}{b}\ \ \sin\frac{n'\pi}{a}y\,dy\,z\,dz.$$

The LHS yields $C_{n,m}\frac{ab}{4}$ and $C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_o(y, z)\sin\frac{n\pi y}{a}\ \sin\frac{n\pi z}{b}\,dy\,dz.$

To proceed further, we must specify $V_o(y, z)$.

Suppose you were to deal with problem with no advance knowledge of sines and cosines. What would you do?

NEXT TIME: Separation of variables in other coordinate systems