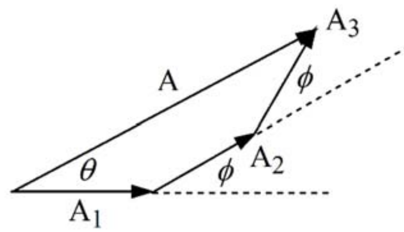
LAST TIME: Finished mirrors and aberrations, more on plane waves

Recall

$$\psi(x,t) = Ae^{i(\mathbf{k}\cdot\mathbf{r}\mp\omega t)}$$

Represents a plane wave having a propagation vector **k** that propagates in any direction with respect to the coordinate axis. We also saw a convenient way of adding waves by what is called the method of phasor addition. A phasor is nothing more than a vector whose length is the amplitude of a wave and whose angle is the phase with respect to the previous wave as the figure shows. Plane waves are certainly not the only kind of wave we wish to consider, but they are very important because any wave may be represented by a linear combination of planes. Spherical waves and cylindrical waves also play an important role in physics. In three dimensions, the wave equation without sources takes the form given by



$$\nabla^2\,\psi(\mathbf{r},t) - \frac{1}{v^2}\frac{\partial^2\psi(\mathbf{r},t)}{\partial t^2} = 0.$$

In Cartesian coordinates, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. ∇^2 is the Laplacian operator and is common in physics. It is not difficult to determine the Laplacian operator in other coordinate systems if you know the transformation equations from one system to the other. In optics, one of the most important wave is the spherically symmetric wave whose wave function is a function of *r* only. In equation form,

$$\psi(r,\theta,\phi) = \psi(r) \text{ only.}$$

Therefore,

$$\nabla^2\psi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi(r)}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\big(r\psi(r)\big).$$

Finally, incorporating the time dependence,

$$\frac{\partial^2}{\partial r^2}\big(r\psi(r)\big) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\big(r\psi(r)\big) = 0.$$

Why are we able to multiply through by *r* in this case?

Therefore,

$$r\psi(r,t) = C_1 f(r - vt) + C_2 g(r + vt)$$

and

$$\psi(r,t) = \frac{C_1 f(r - vt)}{r} + \frac{C_2 g(r + vt)}{r}.$$

The amplitude of a spherical wave changes inversely with *r*. In general, the energy in a wave is proportional to its amplitude squared, so the energy changes inversely with r^2, and the total energy in a sphere of radius *r* does not change. Cylindrical waves do not work out quite so nicely, so we can only write an expression for large *r* or ρ (some books use ρ in place of *r* for cylindrical coordinates) for cylindrical waves. The actual equation is given by

$$\psi(r,t) \cong \frac{A}{\sqrt{r}} e^{i(kr-\omega t)}.$$

For a spherical wave, we may write

$$\psi(r,t) = \frac{A}{r} e^{i(kr-\omega t)}.$$

The dependence on *r* will become clear after we determine how the energy of the wave looks.

Electromagnetic Theory of Light

Maxwell equations (ME) govern electromagnetic behavior. We are going to use them to determine the form of an electromagnetic wave so we can know the properties of a light wave. They may be written in two different forms: the integral form and the differential form. Here they are in both forms for free space.

$$\oiint \mathbf{E}\cdot d\mathbf{S} = \frac{q}{\epsilon_o}; \quad \oiint \mathbf{B}\cdot d\mathbf{S} = 0; \quad \oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}; \quad \oint \mathbf{B}\cdot d\mathbf{l} = \mu_o i + \mu_o \epsilon_o \frac{d\Phi_E}{dt}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_o}; \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0; \quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}; \quad \boldsymbol{\nabla}\times\mathbf{B} = \mu_o\mathbf{J} + \mu_o\epsilon_o\frac{\partial\mathbf{E}}{\partial t}$$

You used the integral form primarily in your introductory course when you were called on to calculate electric and magnetic fields due to charges and currents. The vector operator $\boldsymbol{\nabla}$ depends on the coordinate system in which it is being used, but in Cartesian coordinates it is given by

$$\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.$$

The divergence theorem and Stokes' theorem are used to go back and forth between the integral and differential forms of ME.

$$\oint \mathbf{A}\cdot d\mathbf{l} = \iint (\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{S}; \quad \oiint \mathbf{A}\cdot d\mathbf{S} = \iiint \boldsymbol{\nabla}\cdot\mathbf{A}\, dV;$$

For now, we will not worry how the wave of interest was created, so we can deal directly with the source-free forms of ME. This assumption amounts to setting ρ and $\mathbf{J}$ to zero. Then ME become

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0; \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0; \quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}; \quad \boldsymbol{\nabla}\times\mathbf{E} = \mu_o\epsilon_o\frac{\partial\mathbf{E}}{\partial t}$$

Using the properties of the vector differential operators, we can eliminate either **E** or **B** to see what equations they obey. Here is how the process works starting on the next page.

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial(\nabla \times \mathbf{B})}{\partial t} = -\frac{\partial}{\partial t}\mu_o\epsilon_o\frac{\partial \mathbf{E}}{\partial t} = -\mu_o\epsilon_o\frac{\partial^2\mathbf{E}}{\partial t^2} = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}.$$

$$\nabla^2\mathbf{E} - \mu_o\epsilon_o\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

Therefore, the electric field satisfies a wave equation exactly like what we determined earlier. Now, however, the field is a vector, not a scalar as before. The same equation holds for the magnetic field as well so

$$\nabla^2\mathbf{B} - \mu_o\epsilon_o\frac{\partial^2\mathbf{B}}{\partial t^2} = 0.$$

Notice that to be consistent with our previous wave equation, the speed of the wave must be given by

$$\mu_o\epsilon_o = \frac{1}{v^2} \Longrightarrow v = \frac{1}{\sqrt{\mu_o\epsilon_o}} = c = \text{speed of light in vacuum.}$$

Now that we know ME allow for a wave equation, exactly how is the wave structured? The easiest way to determine the answer to this question is to assume a particular form of the electric field and see how everything else behaves. Therefore, we assume that

$$\mathbf{E} = E_{oy}(x,t)e^{i(kx-\omega t+\epsilon)}\hat{\mathbf{y}};\;\; E_x = E_z = 0$$

These equations for the electric field must satisfy ME if our assumption for the form of **E** can be correct. The curl function is most easily remembered by using the determinant form given by

$$\nabla \times \mathbf{E} = det\begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z\end{bmatrix} = \hat{\mathbf{x}}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{\mathbf{y}}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

Using $E_x = E_z = 0$, we see that

$$\nabla \times \mathbf{E} = \hat{\mathbf{z}}\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}\hat{\mathbf{z}}.$$

Now we use our specific form of the electric field to calculate $\nabla \times \mathbf{E}$ and we obtain

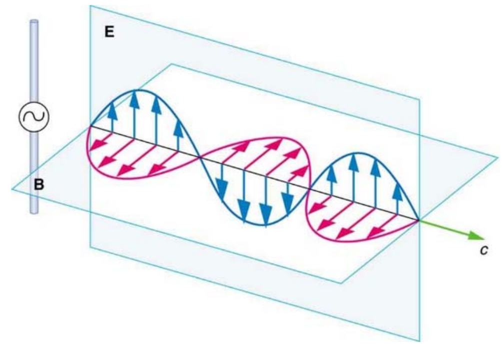
$$\frac{\partial E_y}{\partial x} = ikE_{oy}e^{i(kx-\omega t+\epsilon)} = -\frac{\partial B_z}{\partial t}.$$

The solution for B_z is given by

$$B_z = \frac{-ikE_{oy}}{-i\omega}e^{i(kx-\omega t+\epsilon)} + \text{a function of space only}$$

Therefore $B_z(x,t) = \frac{E_{oy}}{c}e^{i(kx-\omega t+\epsilon)} = B_{oz}e^{i(kx-\omega t+\epsilon)}$ and $\frac{E_{oy}}{B_{oz}} = c$.

Notice now that E and B are perpendicular and they are both perpendicular to the direction of propagation of the wave. Our picture then is **E** ⊥ **B** ⊥ **k**. Notice that **E** and **B** are in phase; i.e., both reach maximum and minimum values at the same time, but the magnitude of **E** is much larger that that of **B**. If we recall the Lorentz force law for charges **F** = q**E** + q**v**×**B**, it seems that **E** should have a much larger effect than **B**. Indeed, this is the case for most materials, so **E** is usually considered the light vector, although **B** must be present for any wave to exist. Here is the algebra showing how it works.



$$\frac{F_{B\,max}}{F_E} = \frac{vB}{E} = \frac{v}{c} \ll 1$$

Otto Weiner did an interesting experiment based on interference that showed experimentally that the electric field did dominate in most materials. EM waves carry energy, momentum, and in some cases angular momentum. Without getting too far afield, here are the exresssions for the energy densities of the electric and magnetic fields.

$$u_E = \frac{\epsilon_o}{2}E^2; \quad u_B = \frac{1}{2\mu_o}B^2.$$

Because $E = cB$, and $c^2 = (\mu_o\epsilon_o)^{-1}, u_B = \frac{1}{2\mu_o}\left(\frac{E}{c}\right)^2 = \frac{\epsilon_o}{2}E^2 = u_E$. This means that the energy in the wave is stored equally between the electric and magnetic fields despite having the elelctric field dominate the forces. The magnitude of the Poynting vector **S** is the power/area crossing a surface and is an important consideration when we study how light behaves at an interface. Recall that power is the energy per time. **S** is given by

$$\mathbf{S} = \frac{1}{\mu_o}\mathbf{E}\times\mathbf{B} = \mathrm{c}^2\epsilon_o\mathbf{E}\times\mathbf{B}.$$

S is parallel to **k** and also perpendicular to both **E** and **B**. Keep in mind that these fields are changing very rapidly because the frequency of green light is on the order of 10^{14} H and w is on the order of 10^{15} rad/s. If we write both fields using cosines, then we have

$$\mathbf{E} = \mathbf{E}_o\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) \text{ and } \mathbf{B} = \mathbf{B}_o\cos(\mathbf{k}\cdot\mathbf{r} - \omega t).$$

Then $\mathbf{S} = c^2\epsilon_o\mathbf{E}_o \times \mathbf{B}_o\cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t)$. Typical optical detectors cannot respond this fast to see the actual wave function, so we need to consider the time average of these types of functions. Therefore, optical detectors give an output that measures $\langle|\mathbf{S}|\rangle$, the time average of the magnitude of the Poynting vector. Let's see how we average harmonic functions. We define the time average of a function as

$$\langle f(t)\rangle_T \equiv \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} f(t)dt.$$

If we use $e^{i\omega t}$ as our function, we can get sine of cosine averages out of it. Therefore,

$$\langle e^{i\omega t}\rangle_T = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} e^{i\omega t}dt = \frac{1}{T}\frac{1}{i\omega}\left[e^{i\omega\left(t+\frac{T}{2}\right)} - e^{i\omega\left(t-\frac{T}{2}\right)}\right]$$

$$= \frac{1}{i\omega T}e^{i\omega t}\left[e^{i\omega\left(\frac{T}{2}\right)} - e^{i\omega\left(\frac{T}{2}\right)}\right] = \frac{1}{i\omega T}e^{i\omega t}\left(2i\sin\frac{\omega T}{2}\right) = e^{i\omega t}\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}} = e^{i\omega t}\text{sinc}\left(\frac{\omega T}{2}\right).$$

Because $e^{i\omega t} = \cos\omega t + i\sin\omega t,$ we now know that

$$\langle\cos\omega t\rangle_T = \cos\omega t\left(\text{sinc}\left(\frac{\omega T}{2}\right)\right) \text{ and } \langle\sin\omega t\rangle_T = \sin\omega t\left(\text{sinc}\left(\frac{\omega T}{2}\right)\right).$$

We need to look at the three cases when $\omega T \gg 1 \Rightarrow T \gg \frac{1}{\omega} \gg \tau =$ period of wave. Here then

$$\langle e^{i\omega t}\rangle_T = 0.$$

If $\omega T \ll 1 \Rightarrow T \ll \frac{1}{\omega} \ll \tau,$ then $\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}} \to 1$ and the average value is just the value of the function at that point. Finally, if $\omega T = 2\pi,\quad \langle e^{i\omega t}\rangle_T = 0$. As part of one of your homework assignments, you will show that

$$\langle\cos^2\omega t\rangle_T = \frac{1}{2}[1 + \text{sinc}\,\omega T\cos 2\omega t]$$

$$\to \frac{1}{2} \text{ if } \omega T \gg 1 \text{ and } = \frac{1}{2} \text{ if } \omega T = 1.$$

In optics, we usually refer to the time-averaged Poynting vector as the irradiance. It is the same as intensity when general EM waves are considered. Therefore,

$$\langle|\mathbf{S}|\rangle_T = c^2\epsilon_o|\mathbf{E}_o\times\mathbf{B}_o|\langle\cos^2\mathbf{k}\cdot\mathbf{r} - \omega t\rangle_T = \frac{c^2\epsilon_o}{2}|\mathbf{E}_o\times\mathbf{B}_o| \text{ for } \omega T \gg 1.$$

Finally,

$$I = \frac{c^2\epsilon_o}{2}\frac{E_o^2}{c} = \frac{c\epsilon_o}{2}E_o^2.$$

What is the expression for the electric field of a HeNe laser operating at 1 mW? Calculate B as well. We will use a beacm diameter of 1 mm with a wavelength of 633 nm. We write the electric field as

$$\mathbf{E} = \mathbf{E}_o e^{i(kz-\omega t)}.$$

$$\langle S\rangle = \frac{(1\times10^{-3})4}{\pi(1\times10^{-3})^2} = \frac{c\epsilon_o}{2}E_o^2 = 1.3\times10^{-3}\,\frac{W}{m^2}$$

$$E_o = \sqrt{\left[\frac{2\langle S\rangle}{c\epsilon_o}\right]} = \left[\frac{2(1.3\times10^{-3})}{(3\times10^{8})(8.85\times10^{-12})}\right]^{1/2} = 980\,\frac{\mathrm{V}}{\mathrm{m}}.$$

Then

$$B_o = \frac{E_o}{c} = 3.3\times10^{-6}\ \mathrm{T}$$

$$k = \frac{2\pi}{\lambda} = 9.9\times10^{6}\ \mathrm{m}^{-1}\ \text{ and } \omega = 3\times10^{15}\ \mathrm{rad/s}$$

Therefore, our wave is

$$\mathbf{E} = 980\,\hat{\mathbf{x}}\,e^{i\left(9.9\times10^{6}z-3\times10^{15}t\right)}\ \mathrm{V/m}$$

and

$$\mathbf{B} = 3.3\times10^{-6}\,\hat{\mathbf{y}}\,e^{i\left(9.9\times10^{6}z-3\times10^{15}t\right)}\ \mathrm{T}$$

In a material such as glass, the irradiance takes on a slightly different to account for being in a material instead of a vacuum. The expression becomes

$$I = \epsilon v\langle E^2\rangle_T$$

Recall that $v = \frac{c}{n} = \lambda_m \nu_m$, but $\nu_m = \nu_{vac}$ and $\frac{c}{\lambda_{vac}} = \frac{c}{n\lambda_m}$. Therefore, $\lambda_m = \frac{\lambda_{vac}}{n}$.

We are now in a position to understand why the amplitude in a spherical wave must vary as (1/r). Because the energy is conserved in the wave, and the area of a sphere increases as r^2, the product of the surface area and the irradiance must be constant. Similarly, for the cylindrical wave, the surface area varies as r, so the amplitude of the wave must vary as $\frac{1}{\sqrt{r}}$.

NEXT TIME: Boundary conditions, Fresnel coefficients (reflection and transmission coefficients), and their applications