LAST TIME: Tensor differentiation, parallel transport, Christoffel symbols, topics in special relativity

Today, we consider matrices, determinants, and Jacobians. I expect that most of you have a solid background in these areas, so we will not spend too much time on these topics. I will consider it primarily a review. Some of the homework will consist of having you prove some of our statements below.

A matrix is a rectangular array of variables, functions, or numbers that satisfies certain rules of operations. It is worthwhile noting that all rank 2 and below tensors may be represented as a matrix. However, it is equally important to mention that not all matrices are tensors. Their elements must satisfy the proper transformation law. Matrices are usually represented by either square brackets or parentheses. Therefore,

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \text{an } m \times n \text{ matrix.}$$

The matrix has *m* rows and *n* columns. Two matrices are equal if all elements are equal, so two matrices can be equal only if they have the same *m* and *n* values. Consider three matrices given by A, B, and C.

A + B = C = B + A
A + 0 = A = 0 + A, where the zero matrix has all elements = 0.
(A + B) + C = A + (B + C)
kA = (kA) = (Ak), *i.e*., each element is multiplied by the scalar and commutes.
Matrix multiplication is a row by column process as follows:

AB = C implies $c_{ij} = a_{ik}b_{kj}$. The number of columns in the first matrix must equal the number of rows in the second matrix and AB $\neq$ BA, so commutation fails. An example is the multiplication of two Pauli matrices.

Let $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so that $\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. But

$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\sigma_1\sigma_3.$$

The unit matrix is given by δ_{ij}. It has ones along the diagonal and zeroes everywhere else. The transpose of a matrix is given by interchanging rows and columns of the matrix. Therefore, if

$$\mathrm{A} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}, \text{then } \mathrm{A}^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}.$$

It is not difficult to show that $(\mathrm{AB})^T = \mathrm{B}^T\mathrm{A}^T$. Similarly, if A and B are square matrices and each has an inverse, then $(\mathrm{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$. Some square matrices are symmetric, and some are skew- or anti-symmetric. We have already seen examples of real, 3×3, symmetric matrices and real, 4×4, anti-symmetric matrices. What are they?

In classical mechanics, matrices are usually real, but matrices need not be real. We may form the conjugate of a complex matrix by replacing every complex element by its complex conjugate. As an example, consider a matrix A given by

$$\mathrm{A} = \begin{pmatrix} a+ib & c-id \\ e-if & g+ih \end{pmatrix}.$$

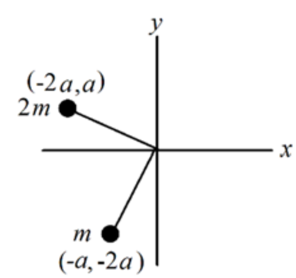
The conjugate matrix is given by

$$\overline{\mathrm{A}} = \begin{pmatrix} a-ib & c+id \\ e+if & g-ih \end{pmatrix}.$$

I use the bar above to indicate the complex conjugate to avoid multiple superscripts. Because the time-dependent Schrödinger equation is inherently complex, dealing with complex matrices is unavoidable. Adjoint matrices are formed by transposing the complex conjugate of a matrix so that $\mathrm{A}^\dagger = (\mathrm{A}^*)^T$. Hermitian matrices (or self-adjoint matrices) are those for which $A = \mathrm{A}^\dagger$. Matrices are labeled as unitary if $\mathrm{U}^\dagger = \mathrm{U}^{-1}$. Unitary matrices have the property that they preserve length for complex vectors which means that they preserve probability amplitudes.

One of the most important problems in mechanics or quantum mechanics is the diagonalization of matrices. As we have already seen in mechanics, the angular momentum is generally not related to the angular velocity by a scalar, but rather by a tensor, the moment of inertia tensor. Recall that this tensor (or matrix) is real and symmetric matrix. These off-diagonal elements occur because we are not using the "right" coordinate system in which to write the angular momentum. Mathematically, we wish to find the values of I such that $I_{ij}\omega_j = I\omega_j$. This is, of course, an eigenvalue problem. Once we find the eigenvalues, we find the eigenvectors. To illustrate the procedure, let's consider a relatively easy system to work on.

The figure shows two point masses, *m* and 2*m*, located at (-*a*,-2*a*) and (-2*a*,*a*), respectively. They are joined to the origin by two massless rods to create a rigid object. (a) Determine the moment of inertia tensor with respect to the origin (0,0). (b) Determine the principal moments of inertia. (c) Determine the principal axis coordinate system. (d) Suppose you attached a mass of 5*m* at the origin to these two masses. Explain why it would or would not affect the moment of inertia tensor. (e) Determine the angular momentum when $\boldsymbol{\omega} = \omega\hat{\mathbf{y}}$.



(a) $I_{xx} = \sum_\alpha m_\alpha\,(y_\alpha^2 + z_\alpha^2) = 2ma^2 + 4ma^2 = 6ma^2$
$I_{xy} = -\sum_\alpha m x_\alpha\, y_\alpha = -(2m)(-2a)(a) - (m)(-a)(-2a) = 2ma^2$.
$I_{yy} = \sum_\alpha m_\alpha\,(x_\alpha^2 + z_\alpha^2) = 2m(4a^2) + ma^2 = 9ma^2$.
$I_{zz} = \sum_\alpha m_\alpha\,(x_\alpha^2 + y_\alpha^2) = 2m[(4a^2 + a^2] + m(a^2 + 4a^2) = 15ma^2$

$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$ because this is a planar object with $z = 0$.
Therefore, the matrix representing the moment of inertia tensor is

$$I_{ij} = ma^2 \begin{bmatrix} 6 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

The eigen value equation is

$$ma^2 \begin{bmatrix} 6-I & 2 & 0 \\ 2 & 9-I & 0 \\ 0 & 0 & 15-I \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 0$$

These types of equations have solutions only if the determinant of the matrix is zero, so

$$\begin{vmatrix} 6-I & 2 & 0 \\ 2 & 9-I & 0 \\ 0 & 0 & 15-I \end{vmatrix} = 0$$

and expanding $(15-I)[(6-I)(9-I)-4] = 0.$ Solving gives $I_3 = 15ma^2$ and

$$54 - 15I + I^2 - 4 = 0.$$

Therefore,

$$I = \frac{15 \pm \sqrt{(225-200)}}{2} = \frac{15 \pm 5}{2} = 5, 10.$$

$$I_1 = 5ma^2, \quad I_2 = 10ma^2, \quad I_3 = 15ma^2.$$

Identifying 1, 2, and 3 with x, y, and z, respectively, we now find the eigenvectors.
For I_3:

$$ma^2 \begin{bmatrix} 6-15 & 2 & 0 \\ 2 & 9-15 & 0 \\ 0 & 0 & 15-15 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -9\omega_x + 2\omega_y \\ 2\omega_x - 6\omega_y \\ 0 \end{bmatrix} = 0$$

so $\omega_x = \omega_y = 0$. $\omega_z = \text{arbitrary} = 1$ to normalize. Therefore, $\hat{\mathbf{e}}_3 = (0, 0, 1)$

For I_2:

$$ma^2 \begin{bmatrix} 6-10 & 2 & 0 \\ 2 & 9-10 & 0 \\ 0 & 0 & 15-10 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -4\omega_x + 2\omega_y \\ 2\omega_x - \omega_y \\ 5\omega_z \end{bmatrix} = 0$$

So $\omega_y = 2\omega_x$. $\omega_z = 0$. Therefore, $\hat{\mathbf{e}}_2 = \frac{1}{\sqrt{5}}(1, 2, 0)$. Note that the two equations are consistent.

For I_1:

$$ma^2\begin{bmatrix}6-5 & 2 & 0\\ 2 & 9-5 & 0\\ 0 & 0 & 15-5\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}=\begin{bmatrix}\omega_x+2\omega_y\\ 2\omega_x+4\omega_y\\ 10\omega_z\end{bmatrix}=0$$

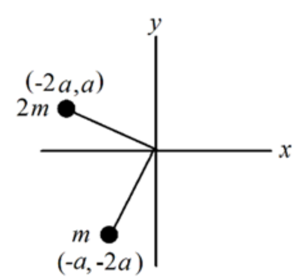
so $\omega_x=-2\omega_y$. $\omega_z=0$. Therefore, $\hat{\mathbf{e}}_1=\frac{1}{\sqrt{5}}(-2,1,0)$. Note again that the two equations are consistent.

Adding any mass at the origin has no effect because I is calculated about the origin.

$$\begin{bmatrix}L_x\\ L_y\\ L_z\end{bmatrix}=ma^2\begin{bmatrix}6 & 2 & 0\\ 2 & 9 & 0\\ 0 & 0 & 15\end{bmatrix}\begin{bmatrix}0\\ \omega\\ 0\end{bmatrix}=ma^2\begin{bmatrix}2\omega\\ 9\omega\\ 0\end{bmatrix}$$

Some discussion of our results:

What is the meaning of these three values of I and three values for unit vectors?



If you had these masses attached to a light bar along the y axis and it was spun as above, what would you feel? Can you see how to calculate the forces if you were in the rotating coordinate system of the rotating masses?

Some useful properties of Hermitian matrices

Have we seen any examples of Hermitian matrices yet in our work? What about

$$\begin{bmatrix}\gamma & -i\beta\gamma & 0 & 0\\ i\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}?$$

Recall that this is the Lorentz transformation matrix when using the $(ict, x,\ y,\ z)$ notation and it is a Hermitian matrix.

The eigenvalues of a Hermitian matrix are real. To see this consider the eigenvalue problem

$\mathrm{Ax}=\lambda\mathrm{x}$, where A is a Hermitian matrix and λ is an eigenvalue. Here is the proof. Multiply by the conjugate transpose of x to get

$$\bar{\mathrm{x}}^T\mathrm{Ax}=\lambda\bar{\mathrm{x}}^T\mathrm{x}$$

and then take the conjugate of each side to get

$$\mathrm{x}^T\bar{\mathrm{A}}\bar{\mathrm{x}} = \bar{\lambda}\mathrm{x}^T\bar{\mathrm{x}}\,.$$

Taking the transpose of each side and remembering to reverse the order gives

$$\bar{\mathrm{x}}^T\bar{\mathrm{A}}^T\mathrm{x} = \bar{\lambda}\bar{\mathrm{x}}^T\mathrm{x}\,.$$

But A is a Hermitian matrix so $\mathrm{A} = \bar{\mathrm{A}}^T$ and $\bar{\mathrm{x}}^T\mathrm{A}\mathrm{x} = \bar{\lambda}\bar{\mathrm{x}}^T\mathrm{x}$. Subtracting the first equation and the last equation gives

$$(\lambda - \bar{\lambda})\bar{\mathrm{x}}^T\mathrm{x} = 0,$$

But $\bar{\mathrm{x}}^T\mathrm{x} \neq 0,$ so $\lambda - \bar{\lambda} = 0$ and λ is real. It is also not hard to prove that eigenvectors belonging to different eigenvalues are orthogonal – the same result for real, symmetric matrices.

Here is an example of diagonalizing a Hermitian matrix. Suppose

$$\mathrm{H} = \begin{bmatrix} 2 & 3-i \\ 3+i & -1 \end{bmatrix}.$$

First, is H Hermitian? Does the complex conjugate of the transpose of H equal to H? Can you have complex diagonal elements?

Write $\begin{vmatrix} 2-\lambda & 3-i \\ 3+i & -1-\lambda \end{vmatrix} = 0$ to get $(2-\lambda)(-1-\lambda) - (3-i)(3+i) = 0.$

Therefore, $\lambda^2 - \lambda - 2 - 9 - 1 = 0 \implies \lambda^2 - \lambda - 12 = 0$ and $\lambda = -3, 4.$

To get the eigenvectors, consider $\begin{bmatrix} 2-\lambda & 3-i \\ 3+i & -1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ and use $\lambda = -3.$

Then $\begin{bmatrix} 2--3 & 3-i \\ 3+i & -1--3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ so $5x + (3-i)y = 0$ and $(3+i)x + 2y = 0.$

These equations may be solved in an ordinary way to get $x = 2$ and $y = -3 - i.$

Using $\lambda = 4$ gives $\begin{bmatrix} 2-4 & 3-i \\ 3+i & -1-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow -2x + (3-i)y = 0 \; and \; (3+i)x - 5y = 0.$

So $y = 2$ and $x = 3 - i$. The two eigenvectors are $\hat{\mathbf{e}}_1 = \frac{(2,-3-i\ )}{\sqrt{14}}$ and $\hat{\mathbf{e}}_2 = \frac{(3-i,2)}{\sqrt{14}}$. Calculating $\hat{\mathbf{e}}_1{}^* \cdot \hat{\mathbf{e}}_2 = 0$ as it must.

In summary of real matrices and complex matrices, Hermitian matrices in the complex plane play the role of real, symmetric matrices in the real world and unitary matrices in the complex plane play the role of orthogonal matrices in the real world. They preserve norms.

Jacobians

For relatively straightforward curvilinear coordinate systems, it is usually not hard to calculate the differential line, surface, and volume elements. In more complicated systems, however, it is useful to have a way to do a formal calculation of the elements. To do this calculation, recall that the volume of a parallelepiped in Cartesian coordinates is just $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, where **A**, **B**, and **C** are vectors along the edges of the parallelepiped. Suppose we have the transformation from *x*, *y*, and *z* to *u*, *v*, and *w*as before. Let vector **A** be along the direction and where *v* and *w* are constant. Then

$$\mathbf{A} = \frac{\partial \mathbf{r}}{\partial u} du = \left(\hat{\mathbf{x}} \frac{\partial x}{\partial u} + \hat{\mathbf{y}} \frac{\partial y}{\partial u} + \hat{\mathbf{z}} \frac{\partial z}{\partial u} \right) du,$$

$$\mathbf{B} = \frac{\partial \mathbf{r}}{\partial v} dv = \left(\hat{\mathbf{x}} \frac{\partial x}{\partial v} + \hat{\mathbf{y}} \frac{\partial y}{\partial v} + \hat{\mathbf{z}} \frac{\partial z}{\partial v} \right) dv,$$

and

$$\mathbf{C} = \frac{\partial \mathbf{r}}{\partial w} dw = \left(\hat{\mathbf{x}} \frac{\partial x}{\partial w} + \hat{\mathbf{y}} \frac{\partial y}{\partial w} + \hat{\mathbf{z}} \frac{\partial z}{\partial w} \right) dw.$$

Therefore, the new volume element is given by

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial x}{\partial w} & \frac{\partial x}{\partial w} \end{vmatrix} du\, dv\, dw.$$

Because we are dealing with a volume, we take the positive sign if the determinant is negative. For a surface element, we just have a 2 × 2 determinant.

NEXT TIME: Sturm-Liouville systems, start partial differential equations solved by separation of variables, and special functions