Physics 6303 Lecture 6 September 10, 2018

LAST TIME: Introduced Cartesian tensors, dyads, outer and inner products, quotient theorem, properties of Kronecker delta and Levi-Civita symbol, and pseudovectors and pseudotensors

We are going to spread our wings and take on the topic of general tensor analysis to see how to build on what we already have done. Recall that Cartesian tensors use only orthogonal transformations that include rotations, reflections, and inversions. It is important to remember that when you use the word tensor to describe an object, you should really say a tensor under what kind(s) of transformations. As a simple example of what I mean, I showed that δ_{ij} *is* a perfectly good Cartesian tensor under rotations. However, under general transformations, it will turn out that is needs to be written as a mixed tensor δ_i^j to qualify as a tensor.

For Cartesian tensors, the transformations are linear transformations in the coordinates that are of the form $\bar{x} = ax + by$ and $\bar{y} = cx + dy$. It is just this linearity between the coordinates that makes the Cartesian tensors relatively easy to handle. For this part of the treatment, I will use barred and unbarred variables to describe the different systems. This is to avoid having both primes and superscripts in the upper part of the variable.

Let's transform from Cartesian coordinates to spherical coordinates using the following equations.

$$x = r\sin\theta\cos\phi\,; \quad y = r\sin\theta\sin\phi\,; \quad z = r\cos\theta.$$

These equations cannot be put into the form of the two equations above that are linear in the coordinates. However, it is possible to create a set of equations that relate their differentials in that form. To do this, write $dx = \sin\theta\cos\phi\ dr + r\cos\theta\ \cos\phi\ d\theta - r\sin\theta\sin\phi\, d\phi,$

$$dy = \sin\theta\sin\phi\ dr + r\cos\theta\ \sin\phi\ d\theta + r\sin\theta\cos\phi\, d\phi,$$

$$dz = \cos\theta\, dr - r\sin\theta\ d\theta.$$

This can be put into the form of a matrix multiplication of linear transformation in the differentials to give

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\ \cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\ \sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

Note that for Cartesian tensors, the transformation between the variables and the differentials is the same. Let's look at a bit simpler case to illustrate something else important. Consider the transformation equations from Cartesian to cylindrical and the inverse.

$$x = \rho\cos\phi; \quad y = \rho\sin\phi; \quad z = \bar{z}.$$

The inverse equations are given by

$$\rho = (x^2 + y^2)^{1/2}; \quad \phi = \tan^{-1}\frac{y}{x}; \quad \bar{z} = z.$$

Now let's connect the differential elements by a transformation matrix.

$$dx = \cos\phi \, d\rho - \rho \sin\phi \;\, d\phi + (0)dz,$$

$$dy = \sin\phi \, d\rho + \rho \cos\phi \;\, d\phi + (0)dz,$$

and

$$dz = (0)dx + (0)dy + d\bar{z}.$$

In matrix form, this gives

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d\rho \\ d\phi \\ d\bar{z} \end{pmatrix}.$$

Now we go the other way by writing $\begin{pmatrix} d\rho \\ d\phi \\ d\bar{z}. \end{pmatrix} = \begin{pmatrix} \frac{x}{(x^2+y^2)^{1/2}} & \frac{y}{(x^2+y^2)^{1/2}} & 0 \\ -\frac{y}{(x^2+y^2)} & \frac{x}{(x^2+y^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix},$

But this matrix can be simplified by using the transforms to a much easier form to deal with and we get

$$\begin{pmatrix} d\rho \\ d\phi \\ d\bar{z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\dfrac{\sin\phi}{\rho} & \dfrac{\cos\phi}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

In passing, please note that if we want to make this expression look more general in terms of $x_i \; and \; \bar{x}_i$, we could have written everything in terms of

$$d\bar{x}_1, d\bar{x}_2, d\bar{x}_3, dx_1, dx_2, \text{and } dx_3, \text{where } \bar{x}_1 = \rho, \qquad \bar{x}_2 = \phi, \text{and } \bar{x}_3 = \bar{z}.$$

Now we need to get serious about notation. When we first started our discussions of coordinate systems, we used superscripts to designate the tangents to the coordinate lines, so we should probably write $d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j$ to transform the differentials (contravariant components). We saw that the gradient operator was associated with the covariant components, so when we consider the partial derivative with respect to $\bar{x}^i$, we would write $\frac{\partial U}{\partial \bar{x}^i} = \frac{\partial U}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial U}{\partial x^j}$. In general, these observations guide us to the following definitions for contravariant and covariant components of a vector or tensor.

$$\bar{V}_i = \frac{\partial x^j}{\partial \bar{x}^i} V_j = \text{covariant components of a vector}$$

and

$$\bar{V}^i = \frac{\partial \bar{x}^i}{\partial x^j} V^j = \text{contravariant components of a vector.}$$

You can understand mathematically how Cartesian vectors and tensors differ from general vector or tensor. In the case of the Cartesian coordinates, each derivative represents the angle between the $\bar{x}^i$ and x^j axes. This means that $\frac{\partial x^j}{\partial \bar{x}^i} = \frac{\partial \bar{x}^i}{\partial x^j}$. However, note that $\partial x/\partial \phi \neq \partial \phi/\partial x$. This, of course, arises from the linear relationships between the coordinates in Cartesian coordinates and the nonlinear relationships in general.

To generalize, for higher rank tensors, we get the following relationships. The results I show below may be extended to any higher order tensor of any type and rank. You just keep adding transformation matrices and subscripts and superscripts.

$$\bar{T}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}T_{kl} = \text{rank 2 covariant tensor},$$

$$\bar{T}^{ijk} = \frac{\partial \bar{x}^i}{\partial x^l}\frac{\partial \bar{x}^j}{\partial x^m}\frac{\partial \bar{x}^k}{\partial x^n}T^{lmn} = \text{rank 3 contravariant tensor}$$

$$T_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^l}\frac{\partial \bar{x}^j}{\partial x^m}\frac{\partial x^n}{\partial \bar{x}^k}T_n^{lm} = \text{mixed tensor} - \text{rank 2 contravariant}, \text{rank 1 covariant}$$

Let's take a look at the Kronecker delta under the transformation for a general tensor. Earlier, I showed that δ_{ij} is an isotropic second rank tensor under rotations (Cartesian tensor). Note, however, what happens in the case of a general transformation. Consider

$$\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}\delta_l^k = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^j} = \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \bar{\delta}_j^i.$$

This shows that in a general transformation, the Kronecker delta is a mixed, isotropic, rank 2 tensor, again illustrating the importance of stating the conditions under which an object has tensor behavior.

Earlier, we also showed the validity of the quotient theorem for Cartesian tensors. What about the quotient theorem in general terms? Let's check it for a special case which generalizes to other cases. Consider that we are given $U_i = T_{ij}V^j$, where U_i and V^j are vectors. We want to show that T_{ij} is a rank 2 covariant tensor. We use the same method as before, but we do not have orthogonal transformations. Therefore,

$$\bar{U}_\alpha = \frac{\partial x^i}{\partial \bar{x}^\alpha}U_i = \frac{\partial x^i}{\partial \bar{x}^\alpha}T_{ij}V^j = \frac{\partial x^i}{\partial \bar{x}^\alpha}T_{ij}\frac{\partial x^j}{\partial \bar{x}^\beta}\bar{V}^\beta = \bar{T}_{\alpha\beta}\bar{V}^\beta \; and \; \bar{T}_{\alpha\beta} = \frac{\partial x^i}{\partial \bar{x}^\alpha}\frac{\partial x^j}{\partial \bar{x}^\beta}T_{ij}.$$

You have probably noticed that we have made frequent use of a process known as contraction, where we sum over an upper and lower index to reduce the rank of a tensor by 2. In general tensors where you need to keep track of covariant and contravariant terms, the contraction is done with respect to an upper and lower index. The most common example of contraction is the scalar product where you write $a_i a^i = a^2$. The product has rank 2 until the contraction when is has rank 0, which is a scalar corresponding to the scalar or dot product.

The metric tensor: Earlier, I referred to the symbol g_{ij} as metric coefficients, but now we are in a position to see that it is a tensor using the quotient theorem. We may now write ds^2 as

$$ds^2 = g_{ij} dx^i \, dx^j.$$

Because ds^2 is a scalar and $dx^i \; and \; dx^j$ are contravariant vectors, we know that g_{ij} is a rank 2 covariant tensor. You may also write $ds^2 = g^{ij} dx_i \, dx_j \; and \; g^{ij}$ is a contravariant tensor that is the inverse of g_{ij}. To show the issue with trying to contract with respect to two superscripts, consider the following problem. You will arrive at the same problem if you try to contract with respect to two subscripts.

$$\bar{v}^i \bar{u}^i = A^i_j v^j A^i_k u^k = \frac{\partial \bar{x}^i}{\partial x^j} v^j \frac{\partial \bar{x}^i}{\partial x^k} u^k.$$

The only way that $A^i_j A^i_k = \delta_{jk}$ is if the A's are orthogonal matrices, which in general they are not. On the other hand, $\bar{v}^i \bar{u}_i = A^i_j v^j A^k_i u_k = \frac{\partial \bar{x}^i}{\partial x^j} v^j \frac{\partial x^k}{\partial \bar{x}^i} u_k = \delta^k_j v^j u_k = v^k u_k = v^j u_j$.

The following expressions are all equivalent.

$$ds^2 = g_{ij} dx^i \, dx^j = g^{ij} dx_i \, dx_j = dx_i dx^j = dx^i dx_j.$$

If you compare these expressions, you see that the metric tensor provides a way to convert contravariant components into covariant components and vice versa. This process is known as index raising or index lowering.

To augment our discussion of general tensors, you may easily prove that tensor algebra follows all the same rules as vector algebra. Addition and subtraction of tensors yields other tensors. Obviously, you can only add or subtract tensors of the same rank, just as you may not add and rank 1 tensor (vector) to a rank 0 tensor (scalar).

We have already seen how to do the outer product of two vectors to give a rank 2 tensor, but you may also do the outer product of two higher rank tensors as follows.

$$A^i_j B^{kl}_m = C^{ikl}_{jm}$$

Contraction corresponds to the inner product.

The metric tensor is of such great importance that a few more comments are in order so you relate the metric tensor to the basis vectors and the coordinate transformations.

Because $ds^2 = d\mathbf{r} \cdot d\mathbf{r}$, we may also write $ds^2 = \mathbf{e}_i dx^i \cdot \mathbf{e}_j dx^j = (\mathbf{e}_i \cdot \mathbf{e}_j) dx^i dx^j$.

Therefore, $g_{ij} = (\mathbf{e}_i \cdot \mathbf{e}_j)$ and it should not be a surprise that $g^{ij} = (\mathbf{e}^i \cdot \mathbf{e}^j)$, so the metric tensor may be generated from either the basis vectors or the dual basis vectors. Because the basis vectors came from the partial derivatives of the transformation equations, it may also be written just using the results we had from our earlier work. See your notes from earlier.

Although tedious to show, the metric tensor may be written as

$$g_{ij} = \left[\left(\frac{\partial \bar{x}^1}{\partial x^i}\frac{\partial \bar{x}^1}{\partial x^j} + \frac{\partial \bar{x}^2}{\partial x^i}\frac{\partial \bar{x}^2}{\partial x^j} + \frac{\partial \bar{x}^3}{\partial x^i}\frac{\partial \bar{x}^3}{\partial x^j}\right)\right] = \frac{\partial \bar{x}^k}{\partial x^i}\frac{\partial \bar{x}^k}{\partial x^j}.$$

Once you know the transformation equations, you may also calculate the metric tensor.

Let's see how the metric tensor plays a role in the fundamental geometry of an ordinary vector **A**. The magnitude of the vector is $A = \sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{A^i\mathbf{e}_i \cdot A^j\mathbf{e}_j} = \sqrt{g_{ij}A^iA^j} = \sqrt{g^{ij}A_iA_j} = \sqrt{A_iA^i}$.

How does the metric tensor play a role in determining the angle between two vectors? Let's use the metric tensor to write the scalar product between two vectors **A** and **B**. Here's how that looks.

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}} = \frac{A_iB^j}{\sqrt{A_iA^i}\sqrt{B_iB^i}} = \frac{g^{ij}A_iB_j}{\sqrt{g^{ij}A_iA_j}\sqrt{g^{ij}B_iB_j}}.$$

The bottom line on all of this is that once you know the transformation equations between the coordinate systems, you also know the basis vectors, the vector components, and the metric tensor.

NEXT TIME: Covariant differentiation, Christoffel symbols, Jacobians, and examples