EXAMPLE: Transforming a Cartesian vector in Cartesian coordinates to a vector in cylindrical coordinates.

Calculate the electric field **E** from a long, straight wire carrying charge density per unit length λ.

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o}\left[\frac{x}{(x^2+y^2)}\hat{\mathbf{x}} + \frac{y}{(x^2+y^2)}\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}\right].$$

Write the transformation equations from lecture 6.

$$x = \rho\cos\phi; \quad y = \rho\sin\phi; \quad z = \bar{z}.$$

The inverse equations are given by

$$\rho = (x^2+y^2)^{1/2}; \quad \phi = \tan^{-1}\frac{y}{x}; \quad \bar{z} = z.$$

Calculate $A^i_j = \frac{\partial \bar{x}^i}{\partial x^j}$, where the barred coordinates are $\rho, \phi,$ and $\bar{z}$. Then use $\bar{E}^i = A^i_j E^j$ to transform the Cartesian components to cylindrical components.

$$A^1_1 = \frac{\partial\rho}{\partial x} = \frac{(1/2)(2x)}{\sqrt{(x^2+y^2)}} = \frac{x}{\rho} = \cos\ \phi; \;\; A^1_2 = \frac{\partial\rho}{\partial y} = \frac{(1/2)(2y)}{\sqrt{(x^2+y^2)}} = \frac{y}{\rho} = \sin\ \phi; \;\; A^1_3 = \frac{\partial\rho}{\partial z} = 0.$$

The trickiest one is $A^2_1 = \frac{\partial\phi}{\partial x}$. Use $\tan\phi = \frac{y}{x} \Rightarrow \sec^2\phi\frac{\partial\phi}{\partial x} = -\frac{y}{x^2} \Rightarrow \frac{\partial\phi}{\partial x} = -\frac{\tan\phi}{\rho\cos\phi}\cos^2\phi = -\frac{\sin\phi}{\rho}$.

Therefore, $A^2_1 = \frac{-\sin\phi}{\rho}$; $A^2_2 = \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{\rho}$; $A^2_3 = \frac{\partial\phi}{\partial z} = 0$.

$A^3_1 = \frac{\partial\bar{z}}{\partial x} = 0$; $A^3_2 = \frac{\partial\bar{z}}{\partial y} = 0$; $\frac{\partial\bar{z}}{\partial z} = 1$

Finally,

$$A^i_j = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ \frac{-\sin\phi}{\rho} & \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{E}^i = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ \frac{-\sin\phi}{\rho} & \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{\lambda}{2\pi\epsilon_o\rho}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

$$\bar{E}^i = \frac{\lambda}{2\pi\epsilon_o\rho^2}\begin{bmatrix} x\cos\phi + y\sin\phi \\ \frac{-x\sin\phi}{\rho} + \frac{y\cos\phi}{\rho} \\ 0 \end{bmatrix}.$$

Eliminate the x and y to obtain

$$\bar{E}^i = \frac{\lambda}{2\pi\epsilon_o\rho^2}\begin{bmatrix} \rho\,(\cos^2\phi + \sin^2\phi) \\ -\rho\cos\phi\sin\phi + \rho\cos\phi\sin\phi \\ 0 \end{bmatrix}$$

$$\bar{E}^i = \frac{\lambda}{2\pi\epsilon_o\rho}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

This problem is trivial, but it shows the power of being able to solve a problem in one coordinate system and transform the solution to another arbitrary coordinate system so long as the transformation equations are known. As an aside to this problem, the values for the B_i^j are given by $\frac{\partial x^j}{\partial \bar{x}^i}$.

$$B_i^j = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The following relationship is true in general as you will prove in the next homework assignment.

$$B^T = A^{-1}$$

Once you have the transformation equations from the barred to the unbarred system and the unbarred system to the barred system, you can transform any covariant, contravariant, or mixed tensor using our definition.