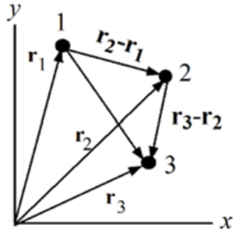
LAST TIME: Examples, reciprocal or dual basis vectors, metric coefficients (tensor), and a few general comments on tensors.

To start this discussion, I will return to two problems that I believe all of you solved in your undergraduate courses. The first one is from intermediate mechanics, the problem of the rotation of an object about a point – not about an axis as you probably recall from your introductory course; the second is from electrodynamics, the problem of the expansion of the electric potential into the different moments. We consider a set of masses, m_α, each located at $\mathbf{r}_\alpha$ and having a velocity $\mathbf{v}_\alpha$. We are interested in how the total angular momentum **L** is related to the angular velocity **ω**.



$$\mathbf{L} = \sum_\alpha \boldsymbol{\ell}_\alpha = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha \text{ where } \mathbf{v}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha.$$

Substituting the value for $\mathbf{v}_\alpha$ and using the BAC – CAB rule gives an expression for $\boldsymbol{\ell}_\alpha$ as

$$\boldsymbol{\ell}_\alpha = m_\alpha[\boldsymbol{\omega}(\mathbf{r}_\alpha \cdot \mathbf{r}_\alpha) - \mathbf{r}_\alpha(\mathbf{r}_\alpha \cdot \boldsymbol{\omega})]$$

Recall that when $\mathbf{r}_\alpha$ is perpendicular to **ω**, which the situation for planar motion, we recover the familiar expression

$$\boldsymbol{\ell}_\alpha = m_\alpha[\boldsymbol{\omega}(\mathbf{r}_\alpha \cdot \mathbf{r}_\alpha)] = (m_\alpha r_\alpha^2)\boldsymbol{\omega}.$$

and

$$\mathbf{L} = \sum_\alpha \boldsymbol{\ell}_\alpha = I\boldsymbol{\omega}, \text{where } I \text{ is a scalar and is given by } \sum_\alpha m_\alpha r_\alpha^2 = \int r^2\, dm.$$

These are the problems you dealt with in your introductory course and so far in this course.

In general, however, $\mathbf{L} = \sum_\alpha \boldsymbol{\ell}_\alpha = \sum_\alpha m_\alpha[\boldsymbol{\omega}(\mathbf{r}_\alpha \cdot \mathbf{r}_\alpha) - \mathbf{r}_\alpha(\mathbf{r}_\alpha \cdot \boldsymbol{\omega})]$.

Now let's consider a case where $\boldsymbol{\omega} = \omega_x\hat{\mathbf{x}} + \omega_y\hat{\mathbf{y}} + \omega_z\hat{\mathbf{z}}$. Under these circumstances, none of the terms disappears. We have two choices as to how to represent **L**. The first choice involves using a lesser known object known as a dyad or dyadic. For this choice, we write **L** as

$$\mathbf{L} = \left(\sum_\alpha m_\alpha r_\alpha^2\right)\boldsymbol{\omega} - \left(\sum_\alpha m_\alpha\, \mathbf{r}_\alpha \mathbf{r}_\alpha\right) \cdot \boldsymbol{\omega}.$$

The juxtaposition of the term $\mathbf{r}_\alpha\mathbf{r}_\alpha$ has not yet been defined, so we need to understand exactly what is meant by this term. We need a short excursion into the world of dyads and dyad algebra. Here are a few rules of dyad algebra. We will eventually learn that the dyad is the outer product of two vectors that turns out to be an excellent way to introduce tensors.

The dyad product of two vectors, **A** and **B** is defined by

$$(\mathbf{AB})\cdot\mathbf{C}=\mathbf{A}(\mathbf{B}\cdot\mathbf{C}).$$

Here are some other dyad algebra rules.

$$(\mathbf{AB})\cdot(c\mathbf{C})=c[(\mathbf{AB})\cdot\mathbf{C}]=c\mathbf{A}(\mathbf{B}\cdot\mathbf{C})$$

$$(\mathbf{AB})\cdot(\mathbf{C}+\mathbf{D})=(\mathbf{AB})\cdot\mathbf{C}+(\mathbf{AB})\cdot\mathbf{D}=\mathbf{A}(\mathbf{B}\cdot\mathbf{C})+\mathbf{A}(\mathbf{B}\cdot\mathbf{D})$$

Essentially, the dyad is a liner vector operator, $\mathbf{F}(\mathbf{C})=(\mathbf{AB})\cdot\mathbf{C}$. It is also a second rand tensor that is formed by the outer product of two vectors. Remember that we are working in a Cartesian coordinate system, so we are really talking about Cartesian tensors. As an aside, technically speaking, the vectors we defined using the rotation of a coordinate system should be called Cartesian vectors, but few people use that terminology. Recall that these vectors transformed according to $\bar{V}_i=\mathcal{R}_{ij}V_j$. I have changed the primes we used earlier to bars to make it easier when we need to use superscripts. Notice that we have not needed to use subscripts and superscripts because in the Cartesian system, there is no need to distinguish between covariant and contravariant components. Before proceeding, let's write out **L** fully in dyad notation.

$$\mathbf{L}=\overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\omega},$$

where $\overleftrightarrow{\mathbf{I}}$ means the dyad given by

$$\overleftrightarrow{\mathbf{I}}=\sum_{\alpha}(m_\alpha r_\alpha^2\overleftrightarrow{\mathbf{1}}-m_\alpha\mathbf{r}_\alpha\mathbf{r}_\alpha),$$

where $\overleftrightarrow{\mathbf{1}}$ is the unit dyad given by $\overleftrightarrow{\mathbf{1}}=\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}$. Because $\mathbf{r}_\alpha$ is a Cartesian vector, we know how it transforms under rotations. This means that the linear vector operator (second rank tensor) transforms as

$$\bar{r}_k\bar{r}_l=\mathcal{R}_{ki}\mathcal{R}_{lj}r_ir_j\text{ or combining }r's\ \bar{T}_{kl}=\mathcal{R}_{ki}\mathcal{R}_{lj}T_{ij}.$$

Had we not introduced the second rank tensor as an outer product of two vectors, we would simply defined the tensor T_{ij} as the last equation. As you might expect, a third-rank tensor is given by

$$\bar{T}_{klm}=\mathcal{R}_{ki}\mathcal{R}_{lj}\mathcal{R}_{mn}T_{ijn}\text{ and so forth.}$$

Every higher rank simply adds another rotation matrix. I have presented a tensor in this form to make one more point that builds on our notion that a scalar is just a number that requires no basis vector, a vector requires a number and one basis vector (one direction), and now a tensor requires one number and two basis vectors (two directions).

Writing out the full moment of inertia dyad gives

$$\overleftrightarrow{\mathbf{I}}=I_{xx}\hat{\mathbf{x}}\hat{\mathbf{x}}+I_{xy}\hat{\mathbf{x}}\hat{\mathbf{y}}+I_{xz}\hat{\mathbf{x}}\hat{\mathbf{z}}+I_{yx}\hat{\mathbf{y}}\hat{\mathbf{x}}+I_{yy}\hat{\mathbf{y}}\hat{\mathbf{y}}+I_{yz}\hat{\mathbf{y}}\hat{\mathbf{z}}+I_{zx}\hat{\mathbf{z}}\hat{\mathbf{x}}+I_{zy}\hat{\mathbf{z}}\hat{\mathbf{y}}+I_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}}.$$

Writing everything out in nice compact tensor notion gives

$$L_i = I_{ij}\omega_j \; with \; I_{ij} = \int \left(r^2\delta_{ij} - r_i r_j\right) dm.$$

Leaving the integral as one over *dm* allows the treatment of linear, areal, or volume mass distributions.

We will return to the moment of inertia tensor later when we do a calculation or two, but for now, let's look at another problem. Frequently, you are asked to prove that an object is a tensor. The quotient theorem is a very powerful mechanism for doing just that. Here is how it works. I have already shown you that the outer product of two vectors is a rank 2 tensor. By extension, it is easy to show that the outer product of any two tensors is also a tensor whose rank is the sum of the ranks of the two tensors involved. Here is the quotient theorem with a proof.

Quotient Theorem: If tensor B is the inner or outer product of C and D, and if C is known to be a tensor, then D is also a tensor. First, I need to remind you about inner products. In fact, the case we just studied, $L_i = I_{ij}\omega_j$ is a contraction. It is the inner product of I_{ij} and ω_j. A contraction reduces the rank by 2, just as the outer product increases the rank by 2.

To prove the theorem for rank 2 tensors (generalization is straightforward), let $B_{ij} = C_{ik}D_{kj}$. Because B_{ij} is a tensor, we may write

$$B'_{nm} = A_{ni}A_{mj}B_{ij} = A_{ni}A_{mj}C_{ik}D_{kj}.$$

But $B'_{nm} = C'_{np}D'_{pm} = A_{ni}A_{pk}C_{ik}D'_{pm}$. Equate the two expressions for B'_{nm} to get

$$A_{ni}A_{mj}C_{ik}D_{kj} = A_{ni}A_{pk}C_{ik}D'_{pm}.$$

Remember that these are Cartesian tensors, so the *A*'s are orthogonal matrices and the transpose is the inverse. We multiply by A_{nq} and use the orthogonal condition ($A_{nq}A_{ni} = \delta_{qi}$) to get

$$A_{mj}C_{qk}D_{kj} = A_{pk}C_{qk}D'_{pm}.$$

We need one more multiplication so we can isolate *C* and *D*. Multiply by A_{mr} and use $A_{mr}A_{mj} = \delta_{rj}$. Therefore, $\left(\delta_{rj}D_{kj} - A_{mr}A_{pk}D'_{pm}\right)C_{qk} = 0$ so $\left(D_{kr} - A_{mr}A_{pk}D'_{pm}\right)C_{qk} = 0$.

Because C_{qk} is arbitrary, $D_{kr} = A_{mr}A_{pk}D'_{pm} = A_{pk}A_{mr}D'_{pm} = A^{-1}_{kp}A^{-1}_{rm}D'_{pm}$.

What are the tensor characteristics of the Kronecker delta and the Levi-Civita symbol? Because we know that $U_i = \delta_{ij}U_j$, where U is a vector, then δ_{ij} is a second rank tensor. However, it is actually more than that. Consider $\delta'_{mn} = A_{mi}A_{nj}\delta_{ij} = A_{mj}A_{nj}\delta_{ij} = \delta_{ij}$ because of the orthogonality of the rotation transformation. Not only is it a rank 2 tensor, it is also called an isotropic tensor, one that has the same values (components) in all coordinate systems. The Levi-Civita symbol is also a tensor because we have seen that $(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk}A_jB_k$. It, too, is an isotropic tensor of rank 3. To prove this statement, we need to develop a technique that uses ϵ_{ijk} to expand a determinant. Why do you think this is important to do? Note that we may write the determinant of A as $\det A = A_{1i}\, A_{2j} A_{3k} \epsilon_{ijk}$. This is equivalent to expansion of the determinant by cofactors, also known as a Laplace development. A second useful formula is

$$\epsilon_{\alpha\beta\gamma} \det A = A_{\alpha i} A_{\beta j} A_{\gamma k} \epsilon_{ijk}.$$

Each of these equations can be worked out by sheer brute force and shown to be true.

Consider the transformation $\epsilon'_{\alpha\beta\gamma} = A_{\alpha i} A_{\beta j} A_{\gamma k} \epsilon_{ijk} = \epsilon_{\alpha\beta\gamma}$ since $\det A = 1$ for all $A's$. So the Levi-Civita symbol is also an isotropic tensor of rank 3. Please remember that for now, we are dealing only with Cartesian tensors (rotated Cartesian coordinate systems). Some of our results will continue to hold for general tensors, whereas others will require a bit of tweaking.

Tensors may also be classified as symmetric or anti (skew) – symmetric. Any rank 2 tensor T_{ij} may be written as $T_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) + \frac{1}{2}\left(T_{ij} - T_{ji}\right)$. If the tensor is higher than rank 2, we say that it is either symmetric or anti-symmetric with respect to two particular indices. You have already seen an example of a real, symmetric tensor in the moment of inertia tensor.

You encounter many other tensors in physics. In electromagnetic theory, for example, you are accustomed to the following constitutive relationships between the various fields:

$$\mathbf{J} = \sigma\mathbf{E};\ \mathbf{D} = \epsilon\mathbf{E};\ \mathbf{B} = \mu\mathbf{H};\ \mathbf{P} = \chi\mathbf{E} \text{ to name a few.}$$

In these equations, $\sigma, \epsilon, \mu,$ and χ are written as scalars (rank 0 tensors). Most of the time, however, particularly when anisotropic materials are involved, these quantities are rank 2 tensors and the constitutive equations become

$$J_i = \sigma_{ij} E_j;\ \ D_i = \epsilon_{ij} E_j;\ \ B_i = \mu_{ij} H_j;\ \ P_i = \chi_{ij} E_j.$$

Note that every one of these is a contraction over j with two other vectors involved, so all of these quantities are tensors by the quotient theorem. In some cases, for certain conditions, the tensors are real and symmetric, and in other cases, especially when magnetic fields are applied, the tensors are anti-symmetric and can be complex. You are likely to see some of these examples in electrodynamics.

You might recognize that one of the identities you have already used involves the product of two isotropic tensors, $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$. The right-hand side is a rank 4 tensor, and the left-hand side is as well. Do you see why?

For the record, there is another common tensor from nonrelativistic electromagnetic theory. Do you know what it is? It shows up in the expansion of the electric potential $V(\mathbf{r})$ for arbitrary charge distributions. Do you remember how the dipole term originated? Recall the multipole expansion for the potential is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' = \frac{1}{4\pi\epsilon_o}\left(\frac{q}{r} + \frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{\sum x_i x_j Q_{ij}}{2r^5} + \cdots\right) \text{ for } r \gg r'.$$

Here, $\mathbf{p} = \int \rho(\mathbf{r}')\,\mathbf{r}' d^3r'$and $Q_{ij} = \int \rho(\mathbf{r}')\big(3x_i x_j - \delta_{ij} r'^2\big)\, d^3r'$. $\mathbf{p}$ is the dipole moment and Q_{ij} is the quadrupole moment tensor. I will do examples of the moment of inertia tensor calculations and the quadrupole moment tensor calculations in the recitation.

Pseudovectors and pseudotensors:

You have undoubtedly heard these terms before in connection with cross products. Orthogonal transformations may also include reflections and inversions, in which case the determinate is -1 instead of 1 as it is with a pure rotation. When the determinate is -1, the “rotation” is sometimes said to be improper. What is the value of the determinant when two reflections are involved? Why? Consider two vectors **U** and **V** and suppose the z-axis is reflected through the x-y plane, so that we have a left-handed coordinate system. The z-components of **U** and **V** must change sign as well. However, notice that the z-component of $\mathbf{U} \times \mathbf{V} = \big(U_x V_y - U_y V_x\big)$ does not change sign. For this reason, the cross product is called a pseudovector. In a similar way, consider the Levi-Civita symbol. Recall $\epsilon_{\alpha\beta\gamma} \det A = A_{\alpha i} A_{\beta j} A_{\gamma k} \epsilon_{ijk}$. Multiply both sides by $\det A$ to get

$$\epsilon_{\alpha\beta\gamma} = \det A \;\; A_{\alpha i} A_{\beta j} A_{\gamma k} \epsilon_{ijk}.$$

For a reflection, $\det A = -1$ so this is not the right transformation equation, and we say that $\epsilon_{\alpha\beta\gamma}$ is a pseudotensor under reflections.

NEXT TIME: General tensor analysis (non-Cartesian tensors), different transformations, metric tensor, and tensor derivatives