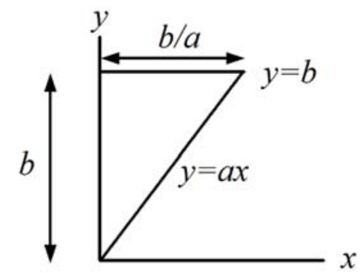
Example of moment of inertia calculation.

Example: Calculate **I** about the origin for the triangle shown. This is a flat object so z=0 and $I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$. This will always be the case for flat objects in the *z* = 0 plane. Let $\sigma = m/A$.



$$I_{xx} = \iint (y^2 + z^2)\sigma dxdy = \int_0^b dy \int_0^{\frac{y}{a}} y^2\, \sigma dx$$

$$= \sigma \int_0^b y^2 dy \int_0^{\frac{y}{a}} 1\, dx = \sigma \int_0^b \left(\frac{y^3}{a}\right) dy = \sigma \frac{b^4}{4a}$$

But

$$\sigma = \frac{2ma}{b^2} \;\; so\ I_{xx} = \frac{mb^2}{2}.$$

$$I_{yy} = \iint (z^2 + x^2)\sigma dxdy = \iint x^2\, \sigma dxdy = \sigma \int_0^b dy \int_0^{\frac{y}{a}} x^2\, dx = \sigma \int_0^b \left(\frac{y^3}{3a^3}\right) dy = \sigma \frac{b^4}{12a^3},$$

So, using the value for σ again,

$$I_{yy} = \frac{mb^2}{6a^2}.$$

It is always a good idea to check dimensions, because here *a* is a slope, not a length.
Finally,

$$I_{zz} = \iint (x^2 + y^2)\, \sigma dxdy = \sigma \int_0^b dy \int_0^{\frac{y}{a}} x^2\, dx + \sigma \int_0^b y^2\, dy \int_0^{\frac{y}{a}} dx.$$

Before we write the final value, let's look at I_{xx} and I_{yy} again. You can see that this is just $I_{xx} + I_{yy}$. This is just the perpendicular axis theorem, so we really do not need to calculate I_{zz} separately. Remember that the perpendicular axis theorem is good only for plane objects. Now, we need only to get I_{xy}.

$$I_{xy} = -\iint xy\, \sigma dxdy = -\sigma \int_0^b y\, dy \int_0^{\frac{y}{a}} x dx = -\sigma \int_0^b \left(\frac{y^3}{2a^2}\right) dy = -\frac{\sigma b^4}{8a^2} = -\frac{mb^2}{4a} = I_{yx}$$

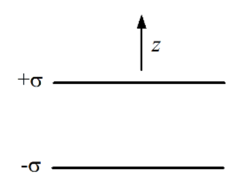
Now write the tensor in matrix form as

$$I = \begin{bmatrix} \frac{mb^2}{2} & -\frac{mb^2}{4a} & 0 \\ -\frac{mb^2}{4a} & \frac{mb^2}{6a^2} & 0 \\ 0 & 0 & \frac{mb^2}{2} + \frac{mb^2}{6a^2} \end{bmatrix}.$$

This is a bit tedious to work with in this form, so let *a* = *b* = 1 for convenience. Then, *I* becomes

$$I = m\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix} = \frac{m}{12}\begin{bmatrix} 6 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Example: Maxwell stress tensor calculation



This is a straightforward calculation designed to show you the power of the Maxwell stress tensor even if the problem is quite easy. Consider a parallel plate capacitor as shown in the figure. Supposed we need to calculate the force on the bottom plate arising from the top plate. We use $\mathbf{F} = q\mathbf{E}$. In using this formula, we must be careful to note that electric field we need is only the electric field from the top plate. Then, we use the charge on the bottom plate only. The electric field at the bottom plate due to the top plate is given by $\mathbf{E} = -\frac{\sigma}{2\epsilon_o}\hat{\mathbf{z}}$.

The total charge on the bottom plate is $-\sigma A$, where A is the area of the plate. Therefore, the force on the bottom plate is given by $\mathbf{F} = A\frac{\sigma^2}{2\epsilon_o}\hat{\mathbf{z}}$. To work this problem using the stress tensor, write

$$T_{ij} = \epsilon_o\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_o}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \text{ and } F_i = \iint T_{ij}n_j\,dS.$$

There is no magnetic field, and the electric field has only one component so we may write T_{zz} as

$$T_{zz} = \epsilon_o\left(E_z^2 - \frac{1}{2}E_z^2\right) = \epsilon_o\frac{E_z^2}{2}.$$

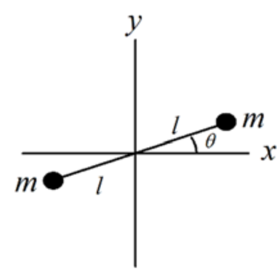
To calculate the force, we enclose the bottom plate in a surface and evaluate T_{zz} over the surface. The field that goes into the calculation is the total field at the surface, which is zero outside and $-\frac{\sigma}{\epsilon_o}\hat{\boldsymbol{z}}$ inside. $n_j = \hat{\boldsymbol{z}}$ so the force is given by

$$F_z = \iint T_{zz}n_z\,dS = \epsilon_oA\frac{E_z^2}{2} = \epsilon_oA\frac{\sigma^2}{2\epsilon_o^2} = A\frac{\sigma^2}{2\epsilon_o}.$$

Although this problem is rather easy, consider the problem of calculating the force on a dielectric boundary when an electromagnetic wave is incident on it. Here, it is much harder to isolate charges and currents, but not hard to calculate the total electric field on each side of the boundary. The stress tensor works perfectly for this application and many other similar ones. We also know that it is a tensor because it is formed by the outer product of two electric field vectors and the Kronecker delta. We will come later to some of the tensors used in relativistic electrodynamics. These are by no means the only places where tensors are used, but they have many uses in physics and engineering.

Example: Rotating a Moment of Inertia Tensor

Consider the moment of inertia of the following system shown to the right. This is a system of discrete masses, so the summation method is used to get



$$\overleftrightarrow{\mathbf{I}} = 2ml^2 \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{ij}$$

We know that one way to get the eigenvalues and the eigenvectors is to diagonalize the matrix and use the eigenvalues to determine the eigenvectors, which make up the principal axis coordinate system. The eigenvalues are the principal moments of inertia in the principal axis coordinate system. According to our figure, we could rotate the tensor by $-\theta$ and bring it back to the x – axis. Formally, that requires us to do the following.

$$\bar{I}_{lm} = \mathcal{R}_{li}\mathcal{R}_{mj}I_{ij},$$

where

$$\mathcal{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ with } \theta \to -\ \theta.$$

Notice that the problem with expressing our problem as a rotation of a tensor is that it does not appear to be a nice matrix multiplication. We can make this much easier by writing

$$\bar{I}_{lm} = \mathcal{R}_{li}I_{ij}\mathcal{R}^T_{jm} \text{ so that}$$

$$\bar{I}_{lm} = 2ml^2 \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{I}_{lm} = 2ml^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can do the actual tensor rotation as follows. $\bar{I}_{lm} = \mathcal{R}_{li}\mathcal{R}_{mj}I_{ij} \Longrightarrow \bar{I}_{11} = \mathcal{R}_{1i}\mathcal{R}_{1j}I_{ij}$

$$\bar{I}_{11} = \mathcal{R}_{11}\mathcal{R}_{11}I_{11} + \mathcal{R}_{11}\mathcal{R}_{12}I_{12} + \mathcal{R}_{11}\mathcal{R}_{13}I_{13} \quad (i = 1, j = 1 \text{ to } 3)$$
$$+\mathcal{R}_{12}\mathcal{R}_{11}I_{21} + \mathcal{R}_{12}\mathcal{R}_{12}I_{22} + \mathcal{R}_{12}\mathcal{R}_{13}I_{23} \quad (i = 2, j = 1\ to\ 3)$$
$$+\mathcal{R}_{13}\mathcal{R}_{11}I_{31} + \mathcal{R}_{13}\mathcal{R}_{12}I_{32} + \mathcal{R}_{13}\mathcal{R}_{13}I_{33} \quad (i = 3, j = 1\ to\ 3).$$

Note that any term with $\mathcal{R}_{13}$ in it is zero. Therefore, only four terms are left and we get

$$\bar{I}_{11} = \mathcal{R}_{11}\mathcal{R}_{11}I_{11} + \mathcal{R}_{11}\mathcal{R}_{12}I_{12} + \mathcal{R}_{12}\mathcal{R}_{11}I_{21} + \mathcal{R}_{12}\mathcal{R}_{12}I_{22}$$
$$\bar{I}_{11} = \cos^2\theta \sin^2\theta - \cos^2\theta \sin^2\theta - \cos^2\theta \sin^2\theta + \cos^2\theta \sin^2\theta = 0.$$