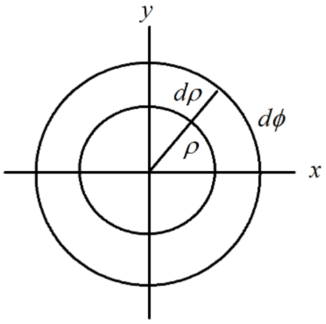
Physics 6303 Lecture 4 August 29, 2018

LAST TIME: Operators in index notation, vector spaces, rotation of the Cartesian system, orthogonal matrices and transformations, curvilinear systems, operators in curvilinear coordinate systems

EXAMPLE: Suppose the magnetic field **B** is given by $\mathbf{B} = B_o(\rho/a^3)(a-\rho)^2\hat{\boldsymbol{\phi}}$, for $0 \le \rho \le a$ and 0 elsewhere. Calculate the current density in the region. We know that $\mathbf{J} = \mu_o^{-1}\nabla \times \mathbf{B}$. **B** is given in cylindrical coordinates. Let's use our previous results to get the curl operator in cylindrical coordinates and then apply it to get **J**. In your homework, you will show that

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}.$$

Get the line element in cylindrical coordinates in terms of $\rho, \phi,$ and z. The figure shows the important issues.



From here, you can see that the line element is given by

$d\mathbf{r} = d\rho\,\hat{\boldsymbol{\rho}} + \rho d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$. Comparing this expression to our original definition in curvilinear coordinates $d\mathbf{r} = h_1\,du_1\hat{\mathbf{e}}_1 + h_2\,du_2\hat{\mathbf{e}}_2 + h_3\,du_3\hat{\mathbf{e}}_3$, we see that $h_1 = 1, h_2 = \rho,$ and $h_3 = 1$.

Therefore, our expression for the curl becomes

$$\nabla \times \mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & A_\phi \rho & A_z \end{vmatrix}$$

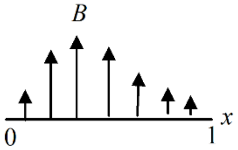
$$= \left(\frac{1}{\rho}\right)\left\{\left[\frac{\partial A_z}{\partial \phi} - \frac{\partial(A_\phi \rho)}{\partial z}\right]\hat{\boldsymbol{\rho}} - \left[\frac{\partial A_z}{\partial \rho} - \frac{\partial A_\rho}{\partial z}\right]\hat{\boldsymbol{\phi}} + \left[\frac{\partial(A_\phi \rho)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\hat{\mathbf{z}}\right\}.$$

To simplify matters, we make some intelligent observations. **B** has only a $\phi-$component and is only a function of ρ. There is only one term that will contribute to the current density, *i.e.*, $\left(\frac{1}{\rho}\right)\frac{\partial(A_\phi \rho)}{\partial \rho}\hat{\mathbf{z}}$. Then $\rho B_\phi = B_o(\rho^2/a^3)(a-\rho)^2$ so we evaluate

$$\frac{B_o}{\rho a^3}\frac{\partial}{\partial \rho}[\rho^2(a-\rho)^2] = \frac{B_o}{\rho a^3}[2\rho(a-\rho)^2 - 2\rho^2(a-\rho)]\hat{\mathbf{z}}.$$

Factoring the term in brackets and putting in the remaining terms, we get

$$\mathbf{J} = \frac{2B_o}{\mu_o a^3}(a-\rho)(a-2\rho)\hat{\mathbf{z}}.$$



This result deserves a bit of discussion. If $\rho < a/2$, then **J** is in the positive z direction. However for $a/2 < \rho < a$, **J** is in the negative z direction. Let's look at the **B** field as a function of ρ for $\phi = 0$ or along the x axis. By symmetry, we can get the structure of the field everywhere else. Along the x axis, we can take $a = 1$ and note that **B** is in the y-direction, and its structure is determined by $x(1-x)^2$. Because **J** is proportional to curl **B**, you can see that when you calculate $\oint \mathbf{B} \cdot d\boldsymbol{\ell}$ for values smaller than B_{max}, it is negative, but for values greater than , it is positive. This is consistent with our calculation.

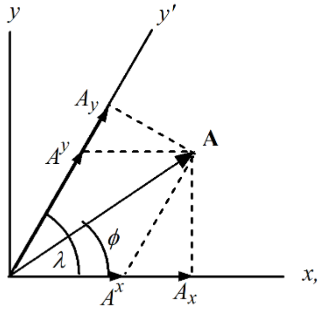
EXAMPLE

Suppose we wished to solve a spherically symmetric potential distribution problem using Laplace's equation from electrostatics, $\nabla^2\Phi = 0$. We need the Laplacian operator in spherical coordinates. We know that we need to calculate the divergence of the gradient, $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\Phi$. Therefore,

$$\boldsymbol{\nabla} \cdot \left[\frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3}\right]$$
$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right].$$

Now, we just need to find the line element to get h_1, h_2, and h_3. $d\mathbf{r} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}} +$. Therefore, $h_1 = 1, h_2 = r,$ and $h_3 = r\sin\theta$. It is straightforward, but tedious, to plug in the values to get the usual expression for the Laplacian in spherical coordinates. We will come to the solution of this equation when we study special functions.

<u>Reciprocal or dual basis vectors</u>



Recall the short example I mentioned at the very beginning of our discussion of coordinate systems. Here it is as a reminder. Recall $\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3$. The superscripted components are represented using the subscripted basis vectors, but the subscripted components are also represented using the same subscripted basis vectors. The superscripted basis vectors are known as the reciprocal or dual basis vectors. It is not hard to prove that the definition we gave originally of the superscripted and subscripted basis vectors are reciprocal basis vectors. Here is the definition of dual (reciprocal) basis vectors.

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3}.$$

These equations ensure that reciprocal basis vectors have two characteristics. They are:
1. $\mathbf{e}^1$ is perpendicular to $\mathbf{e}_2$ and $\mathbf{e}_3$. Permuting the numbers gives similar results for the other two reciprocal vectors.

2. The triple scalar product in the denominator ensures that $\mathbf{e}^1 \cdot \mathbf{e}_1 = 1$ with similar results for the other two sets of basis vectors.

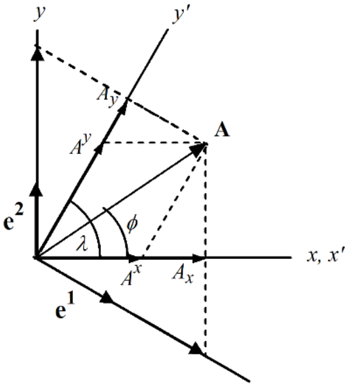
For the case we are dealing with, let's assume that the *z* and *z′* axes align. Suppose the *x*-*x′*-axis is the 1-axis, the *y′*-axis is the 2-axis, and the *z*-*z′*-axis is the 3-axis. Construct the reciprocal basis vectors.

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3} = \frac{(\hat{\mathbf{x}} \cos\lambda + \hat{\mathbf{y}} \sin\lambda) \times \hat{\mathbf{z}}}{\hat{\mathbf{x}} \cdot [(\hat{\mathbf{x}} \cos\lambda + \hat{\mathbf{y}} \sin\lambda) \times \hat{\mathbf{z}}]} = \frac{-\hat{\mathbf{y}} \cos\lambda + \hat{\mathbf{x}} \sin\lambda}{\hat{\mathbf{x}} \cdot [-\hat{\mathbf{y}} \cos\lambda + \hat{\mathbf{x}} \sin\lambda]}$$

$$= \frac{-\hat{\mathbf{y}} \cos\lambda + \hat{\mathbf{x}} \sin\lambda}{\sin\lambda},$$

$$\mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3} = \frac{\hat{\mathbf{z}} \times \hat{\mathbf{x}}}{\sin\lambda} = \frac{\hat{\mathbf{y}}}{\sin\lambda}, \text{and } \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3} = \frac{\hat{\mathbf{x}} \times (\hat{\mathbf{x}} \cos\lambda + \hat{\mathbf{y}} \sin\lambda)}{\sin\lambda} = \hat{\mathbf{z}}.$$

Now we redraw the entire system with the reciprocal basis vectors to see how vector **A** projects onto the reciprocal basis vectors. Now that we have both components projected onto the correct basis vectors, vector addition is preserved for both. It is important to notice that the dual basis vectors are no longer unit vectors even though we started with unit vectors in the original system. I have spent some time on this problem so that you become comfortable with the notion of covariant and contravariant components of a vector, as these ideas can be important in the study of special and general relativity as well as many other topics in physics.



The Metric Coefficients (Tensor)

Earlier, we introduced the idea of the line element for each of the coordinate systems as a way to help determine the gradient and other vector operators. There is another very important use for the line element, as it forms the basis for constructing the metric for the coordinate system.

We consider two points in a particular coordinate system that are separated by an infinitesimal distance *ds*. The simplest example is our friendly rectangular Cartesian coordinate system where $d\mathbf{r} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz, so\ ds^2 = d\mathbf{r} \cdot d\mathbf{r} = dx^2 + dy^2 + dz^2$. In all orthogonal coordinate systems, since the basis vectors are orthogonal, there will be no cross terms such as $dx\ dy$. If the terms are written in matrix form, there are never any off-diagonal elements in the matrix. To put this in a more general form, recall that we used

$$d\mathbf{r} = h_1 du_1 \hat{\mathbf{e}}_1 + h_2 du_2 \hat{\mathbf{e}}_2 + h_3 du_3 \hat{\mathbf{e}}_3.$$

With all the *h*'s equal 1 as well as the other connections, you can see how the general expression reduces to the familiar one for Cartesian coordinates.
We have also worked out the values for the *h*'s for cylindrical and spherical coordinates, so their metric coefficients are not difficult to get. It is not necessary to understand all of the tensor

properties of the metric in order to see its usefulness. Let's be a bit more general and see just how we can represent the distance *ds*. We have just learned about the ways in which to write a vector, so note that $d\mathbf{r} = \mathbf{e}_i dx^i$, where I am using the index notation for convenience. Note here that $\mathbf{e}_i$ is not necessarily a unit vector. We know that we may also write $d\mathbf{r} = \mathbf{e}^i dx_i$. You can see that multiple ways of writing ds^2 exist. Here are the three most common ways.

$$ds^2 = \mathbf{e}_i dx^i \cdot \mathbf{e}_j dx^j = \left(\mathbf{e}_i \cdot \mathbf{e}_j\right) dx^i dx^j = g_{ij} dx^i dx^j,$$

$$ds^2 = \mathbf{e}^{\boldsymbol{i}} dx_i \cdot \mathbf{e}^{\boldsymbol{j}} dx_j = (\mathbf{e}^{\boldsymbol{i}} \cdot \mathbf{e}^{\boldsymbol{j}}) dx_i dx_j = g^{ij} dx_i dx_j, \text{ and}$$

$$ds^2 = \mathbf{e}_i dx^i \cdot \mathbf{e}^{\boldsymbol{j}} dx_j = (\mathbf{e}_i \cdot \mathbf{e}^{\boldsymbol{j}}) dx^i dx_j = dx^i dx_i.$$

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The last step comes from the definition of reciprocal or dual basis vectors $(\mathbf{e}_i \cdot \mathbf{e}^{\boldsymbol{j}}) = 1$ if $i = j$ and zero otherwise. In three-dimensional space, it is clear that g may be written as a matrix, but we have not shown that it is a tensor. Before we leave this topic and proceed to a discussion of tensors, let's write out in more detail some of the terms I have written more concisely before. We have identified the scale factors simply by calculating the line element, but there is another way by using the transformation equations. Here it is for completeness. Recall

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3$$

$$= h_1\, du_1 \hat{\mathbf{e}}_1 + h_2\, du_2 \hat{\mathbf{e}}_2 + h_3\, du_3 \hat{\mathbf{e}}_3,$$

Recall that $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Note that

$$\hat{\mathbf{e}}_1 = \frac{\dfrac{\partial \mathbf{r}}{\partial u_1}}{\left|\dfrac{\partial \mathbf{r}}{\partial u_1}\right|} \text{ so } h_1 = \left|\frac{\partial \mathbf{r}}{\partial u_1}\right| = \left|\hat{\mathbf{x}} \frac{\partial x}{\partial u_1} + \hat{\mathbf{y}} \frac{\partial y}{\partial u_1} + \hat{\mathbf{z}} \frac{\partial z}{\partial u_1}\right|$$

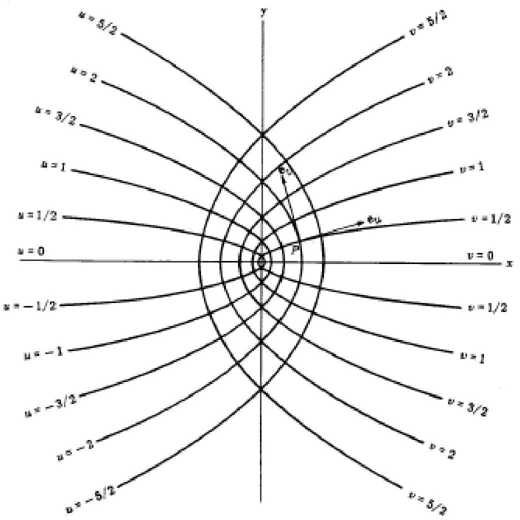
$$= \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}.$$

Furthermore, you can also see that

$$\hat{\mathbf{e}}_1 = \frac{1}{h_1}\left[\hat{\mathbf{x}} \frac{\partial x}{\partial u_1} + \hat{\mathbf{y}} \frac{\partial y}{\partial u_1} + \hat{\mathbf{z}} \frac{\partial z}{\partial u_1}\right].$$

Similar expressions hold for the other components. The main point here is that you can also obtain the expressions for the values of *h*'s and the values for the unit vectors from the transformation equations between the two coordinate systems.

EXAMPLE: Determine the scale factors, metric coefficients, and the volume element for parabolic cylindrical coordinates. The surfaces are confocal parabolas with a common axis and a plane *z* =*z* parallel to the *x*-*y* plane. The transformation equations are given by



$x = \frac{1}{2}(u^2 - v^2); y = uv; z = z,$ where $-\infty < u < \infty, v \geq 0,$ and $-\infty < z < \infty.$

Evaluate partial derivatives to get

$$\frac{\partial x}{\partial u_1} = \frac{\partial x}{\partial u} = u; \ \frac{\partial y}{\partial u_1} = \frac{\partial y}{\partial u} = v; \ \frac{\partial z}{\partial u_1} = \frac{\partial z}{\partial u} = 0.$$

Therefore, $h_u = \sqrt{(u^2 + v^2)} = h_v \; and \; h_z = 1.$

To obtain the metric coefficients, we use either of two methods. First, use

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

with $dx = udu - vdv, dy = udv + vdu, and \; dz = dz.$ Therefore,

$$(dx)^2 = \; u^2(du)^2 - 2uv\; du\, dv + v^2(dv)^2$$

and

$$(dy)^2 = \; u^2(dv)^2 + 2uv\; du\, dv + v^2(du)^2.$$

Adding the terms and grouping appropriately gives

$$(ds)^2 = (u^2 + v^2)(du)^2 + (u^2 + v^2)(dv)^2 + (dz)^2.$$

We may also go back and use the line element to get

$$(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2 = (u^2 + v^2)(du)^2 + (u^2 + v^2)(dv)^2 + (dz)^2.$$

Introduction to Tensors

General comments: There are many types of tensors, and it is important to make some distinctions before we get to deep into the subject. In general, we think of tensors as objects that behave in a certain way under certain types of transformations. We have already seen that vectors are defined as objects that behave the same way as a point behaves under rotations. These rotations were orthogonal transformations because they transform the components from one rectangular Cartesian coordinate system to another rotated Cartesian coordinate system. It is always important to remember the vector does not change – only its components change.

I also mentioned affine transformations that can reflect, invert, stretch, and translate. We also looked at curvilinear (orthogonal) coordinate systems and nonorthogonal coordinate systems.

For the first part of our discussion, we focus on what are called Cartesian tensors because we will deal with rectangular Cartesian coordinate systems.

NEXT TIME: Tensors, dyads and dyadics, examples