EXPAND DUAL BASIS VECTORS: Operators in index notation, vector spaces, rotation of the Cartesian system, orthogonal matrices and transformations, curvilinear systems, operators in curvilinear coordinate systems

As part of homework set 2, you will show that

$$A_i A^i = A^2.$$

Because $\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3$, the expression $A_i A^i = A^2$ imposes conditions on the basis sets $\boldsymbol{e}_i$ and $\boldsymbol{e}^j$. The lack of any cross terms in the above expression means that

$$\boldsymbol{e}_i \cdot \boldsymbol{e}^j = \delta_i^j.$$

This expression for the two basis vectors is really the definition of a dual basis vector system. You may note that I have used a subscript and superscript for the Kronecker delta, but its meaning is still the same as our old friend δ_{ij}. This is simply the better way of writing it when covariant and contravariant components are present. More about this a couple of lectures from now after we introduce tensors. Let's see how to construct the dual (reciprocal) basis to $\boldsymbol{e}_i$. Because all the cross terms vanish, the following relationship must be true.

$$\boldsymbol{e}^1 = m(\boldsymbol{e}_2 \times \boldsymbol{e}_3).$$

We also know that $\boldsymbol{e}_1 \cdot \boldsymbol{e}^1 = 1,$ so $m\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3) = 1$ and $m = \frac{1}{\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)}$. Therefore, we may write the dual basis vectors as

$$\boldsymbol{e}^1 = \frac{(\boldsymbol{e}_2 \times \boldsymbol{e}_3)}{\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)}, \boldsymbol{e}^2 = \frac{(\boldsymbol{e}_3 \times \boldsymbol{e}_1)}{\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)}, \text{and } \boldsymbol{e}^3 = \frac{(\boldsymbol{e}_1 \times \boldsymbol{e}_2)}{\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)}.$$

We may do the same operations to get the other relationships as follows.

$$\boldsymbol{e}_1 = \frac{\boldsymbol{e}^2 \times \boldsymbol{e}^3}{\boldsymbol{e}^1 \cdot (\boldsymbol{e}^2 \times \boldsymbol{e}^3)}, \boldsymbol{e}_2 = \frac{\boldsymbol{e}^3 \times \boldsymbol{e}^1}{\boldsymbol{e}^1 \cdot (\boldsymbol{e}^2 \times \boldsymbol{e}^3)}, \text{and } \boldsymbol{e}_3 = \frac{\boldsymbol{e}^1 \times \boldsymbol{e}^2}{\boldsymbol{e}^1 \cdot (\boldsymbol{e}^2 \times \boldsymbol{e}^3)}.$$

Note that in each case the denominator is the volume of the parallelepiped created by the basis system.