LAST TIME: Coordinate system construction, covariant and contravariant vector components, basics vector review, gradient, divergence, curl, and Laplacian operators

Let's stay with rectangular Cartesian coordinates for now and consider the gradient, divergence, curl, and Laplacian.

For a vector field $\mathbf{A}(x, y, z) = A_x\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$, the divergence and curl are given by

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

and

$$\boldsymbol{\nabla}\times\mathbf{A} = \det\begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{bmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{\mathbf{y}}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).$$

What is the physical significance of each of these? The gradient operator acts on a scalar, but the Laplacian operator may act on either a scalar or a vector. If it acts on a vector, it acts on each of the components of the vector; otherwise, acting on a scalar is obvious.

Because $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$, the operator becomes $\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)$ so

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Example

Consider a scalar field given by $U(x,y,z) = xy^2 + x^3yz + 2xyz$. Calculate $\boldsymbol{\nabla}U(x, y, z) = \mathbf{B}(x, y, z)$.

$$\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(xy^2 + x^3yz + 2xyz) =$$

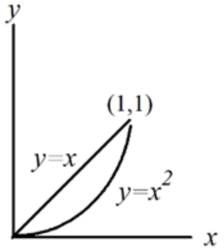
$$\hat{\mathbf{x}}(y^2 + 3x^2yz + 2yz) + \hat{\mathbf{y}}(2xy + x^3z + 2xz) + \hat{\mathbf{z}}(x^3y + 2xy) = \mathbf{B}(x, y, z).$$

Now let's calculate

$$\boldsymbol{\nabla}\times\mathbf{B}(x, y, z) = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - \hat{\mathbf{y}}\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) \text{ to get}$$

$$\hat{\mathbf{x}}[(x^3 + 2x) - (x^3 + 2x)] - \hat{\mathbf{y}}[(3x^2y + 2y) - (3x^2y + 2y)] + \hat{\mathbf{z}}[(2y + 3x^2z + 2z) - (2y + 3x^2z + 2z)] = \mathbf{0}.$$

We knew this ahead of time because of what identity? What does **B** represent?



Finally, let's do the line integral of **B** along two different paths. To make the algebra a bit easier, let's do the integral along the paths shown in the figure with $z = 0$.

Therefore, $\mathbf{B}(x, y, 0) = y^2\hat{\mathbf{x}} + 2xy\hat{\mathbf{y}}$ so the line integral is written as

$\int_i^f \mathbf{B}\cdot \mathrm{d}\mathbf{r} = \int_i^f \left(B_x dx + B_y dy\right)$. Now go along the path $y = x$ where $dx = dy$. The key to doing line integrals is to express everything in terms of either x or y so we know the initial and final values over which to do the integral. Making a direct substitution into the line integral equation gives $\int_i^f \left(B_x dx + B_y dy\right) = \int_i^f (y^2dx + 2xydy)$. We can use the equation of the path to get $\int_0^1 (x^2dx + 2x^2dx) = \int_0^1 (3x^2dx) = x^3|_0^1 = 1$. If you do not like using the x variable, then you can use the y variable just as well.

$$\int_i^f (y^2dx + 2xydy) = \int_0^1 (y^2dy + 2y^2dy) = 1.$$

Now look at the parabolic path using either variable. Here $dy = 2xdx$.

$$\int_i^f (y^2dx + 2xydy) = \int_0^1 (x^4dx + 2xx^2 2xdx) = \int_0^1 (x^4dx + 4x^4dx) = \int_0^1 5x^4dx = x^5|_0^1$$
$$= 1.$$

You may also do the integrals using a parametric representation of the function. Piecing things together, we started with a scalar that could represent a potential energy function, calculated its gradient (the negative of the associated force), evaluated the curl to be zero as it should be for a conservative force, and then found that the line integral was the same for two different paths. There should be no surprise in any of these results. Are there other line integrals? Of course, but they are probably not as physically transparent as the line integral we just did. $\int \phi\, d\mathbf{r}$ and $\int \boldsymbol{V} \times d\mathbf{r}$ are two such integrals. Please look at your book's Example 3.7.1 on page 160.

Let's now look at the different kinds of surface integrals we might do. Here are four common ones.

$$\int f(x, y, z)\hat{\mathbf{n}}\, dS;\quad \int \mathbf{V}\cdot\hat{\mathbf{n}}\, dS;\quad \int \mathbf{V}\times\hat{\mathbf{n}}\, dS;\quad \int f(x, y, z)\, dS$$

Which one is the most common one for applications in physics?

Example

Evaluate $\iint \mathbf{A}\cdot\hat{\mathbf{n}}\, dS$ where $\mathbf{A} = 18\mathrm{z}\,\hat{\mathbf{x}} - 12\,\hat{\boldsymbol{y}} + 3y\,\hat{\mathbf{z}}$ and S is that part of the plane $2x + 3y + 6z = 12$ that lies in the first quadrant. Since dS is arbitrary in general, it is easier to write its projection onto the x-y plane and do the integral over x and y. This means that $dS = \frac{dxdy}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{z}}|}$. Then, we may do the integral over the usual coordinates. How do we find $\hat{\mathbf{n}}$? One of your homework problems has you show that the gradient of the surface is perpendicular to the surface. Therefore,

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\nabla}(2x+3y+6z)}{(4+9+36)^{\frac{1}{2}}} = \frac{1}{7}(2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 6\hat{\mathbf{z}}).$$

This means that $dS = \left(\frac{7}{6}\right) dxdy,$ so now we have to calculate $\mathbf{A}\cdot\hat{\mathbf{n}}$ and then set the limits of integration. $\mathbf{A}\cdot\hat{\mathbf{n}} = (18z\,\hat{\mathbf{x}} - 12\,\hat{\mathbf{y}} + 3y\,\hat{\mathbf{z}}) \cdot \left(\frac{1}{7}\right)(2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 6\hat{\mathbf{z}}) = \left(\frac{1}{7}\right)(36z - 36 + 18y).$

We do not want z in the integral, so we eliminate it by using the equation for the surface. Therefore,

$$z = \left(\frac{1}{6}\right)(12 - 2x - 3y)\ and\ \mathbf{A}\cdot\hat{\mathbf{n}} = \left(\frac{1}{7}\right)\left(36\left(2 - \frac{1}{3}x - \frac{1}{2}y\right) - 36 + 18y\right)$$

$$= \left(\frac{1}{7}\right)(36 - 12x).\ \ \text{Therefore,}$$

$$\iint \mathbf{A}\cdot\hat{\mathbf{n}}\, dS = \iint \left(\frac{1}{7}\right)(36 - 12x)\left(\frac{7}{6}\right) dxdy = \iint (6 - 2x)\, dxdy.$$

We are working in the z = 0 plane, so this becomes an ordinary double integral. Setting z = 0 in the expression 2x + 3y +6z = 12 gives 2x + 3y= 12 or $y = 4 - \frac{2}{3}x.$ Finally, the double integral is given by

$$\int_{x=0}^{6}\int_{y=0}^{4-\frac{2}{3}x} (6 - 2x)\, dydx = \int_{x=0}^{6}\left(24 - 12x + \frac{4x^2}{3}\right) dx = \left(24x - 6x^2 + \frac{4x^3}{9}\right)\Big|_0^6 = 24.$$

How would we find the equation of the plane if we were not given the equation? Recall that 3 points determine a plane. What condition on the points exists? Therefore, construct vectors that determine the plane, determine the normal to the plane, and then pick an arbitrary point to use the scalar produce.

$\mathbf{r}_1 = x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + z_1\hat{\mathbf{z}},\ \mathbf{r}_2 = x_2\hat{\mathbf{x}} + y_2\hat{\mathbf{y}} + z_2\hat{\mathbf{z}},\ \mathbf{r}_3 = x_3\hat{\mathbf{x}} + y_3\hat{\mathbf{y}} + z_3\hat{\mathbf{z}},$ and $r = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.$

Therefore, $(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1)$ is a vector perpendicular to the plane. Because $(\mathbf{r} - \mathbf{r}_1)$ lies in the plane, it is perpendicular to the cross product. We can make good use of the triple scalar product to see that $(\mathbf{r} - \mathbf{r}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1) = 0$ gives the equation of the plane as

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

Some useful integral theorems have great importance in physics. At the beginning, we wrote Maxwell's equations in differential form as

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho; \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0; \quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}; \quad \boldsymbol{\nabla}\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$$

You should recall that you used the integral form of these equations along with high symmetry situations to calculate electric fields and magnetic fields. Let's consider two of the most important integral theorems. What are they?

Divergence theorem:

$$\begin{aligned}&\iiint \boldsymbol{\nabla}\cdot\mathbf{A}\,dxdydz\\ &\qquad= \iiint\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)dxdydz\\ &\qquad= \iiint dA_x\,dydz+\iiint dA_y\,dxdz+\iiint dA_z\,dxdy\\ &\qquad= \iint A_x\,dydz+\iint A_y\,dxdz+\iint A_z\,dxdy=\iint \mathbf{A}\cdot\hat{\mathbf{n}}\,dS.\end{aligned}$$

Applying this theorem to Gauss's law in differential form and using free space gives

$$\iiint \boldsymbol{\nabla}\cdot\mathbf{E}\,dV=\iiint\frac{\rho}{\epsilon_o}dV\ so\ \iint \mathbf{E}\cdot\hat{\mathbf{n}}\,dS=\frac{q}{\epsilon_o}.$$

We may also apply Stokes' theorem to Faraday's law to get the Maxwell form of Ampere's law.

$$\iint(\boldsymbol{\nabla}\times\mathbf{A})\cdot\hat{\mathbf{n}}dS=\oint\mathbf{A}\cdot d\mathbf{r}.$$

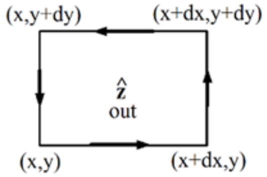
Switch to free space versions to get

$$\boldsymbol{\nabla}\times\mathbf{B}=\mu_o\mathbf{J}+\mu_o\epsilon_o\frac{\partial\mathbf{E}}{\partial t}.$$

Now use Stokes' theorem to get

$$\iint\boldsymbol{\nabla}\times\mathbf{B}\cdot\hat{\mathbf{n}}dS=\iint\mu_o\mathbf{J}\cdot\hat{\mathbf{n}}dS+\mu_o\epsilon_o\iint\frac{\partial\mathbf{E}}{\partial t}\cdot\hat{\mathbf{n}}dS=\mu_o(I+I_d)=\oint\mathbf{B}\cdot\mathrm{d}\boldsymbol{\ell},$$

where I_d is the displacement current.



Proof of Stokes' theorem:
Consider the figure to the right. We need to prove

$$\iint\boldsymbol{\nabla}\times\mathbf{A}\cdot\hat{\mathbf{n}}dS=\oint\mathbf{A}\cdot\mathrm{d}\mathbf{r}.$$

Let $\mathbf{A}=A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}}$ and calculate the curl of **A**.

$$\boldsymbol{\nabla}\times\ \mathbf{A}(x,y,z)=\hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\hat{\mathbf{y}}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+\hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right).$$

Now

$$\iint\boldsymbol{\nabla}\times\mathbf{A}\cdot\mathbf{n}dS=\iint\left[\hat{\mathbf{x}}\cdot\hat{\mathbf{n}}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\hat{\mathbf{y}}\cdot\hat{\mathbf{n}}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+\hat{\mathbf{z}}\cdot\hat{\mathbf{n}}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\right]dS.$$

In general, the surface normal will have components along each direction, so we pick one, $\hat{\mathbf{n}}=\hat{\mathbf{z}}$, to get

$$\iint(\boldsymbol{\nabla}\times\mathbf{A})_z\cdot\mathbf{n}dS=\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)dxdy,\text{where S is a small rectangle.}$$

Now we compute the right-hand side using the standard line integral around the same small rectangle to get

$$\oint \mathbf{A}\cdot d\mathbf{r} = A_x(x,y)dx + A_y(x+dx,y)dy - A_x(x,y+dy)dx - A_y(x,y)dy$$
$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)dxdy,$$

so the theorem is proved by considering the other normal components and adding.
For a two-dimensional surface, Stokes' theorem is just Green's theorem in the plane and becomes

$$\iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dxdy = \oint f dx + g dy$$

In your textbook, you can find numerous other integral formulas, some of which are more useful than others.

Index formulation of vectors

As long as we stay in an orthogonal coordinate system, we have already seen that is makes no difference whether we use subscripts or superscripts, so we will stay with subscripts until it matters. We usually represent a vector **A** by giving its components A_i, where it is understood that i ranges from 1 to 3 for 3 dimensions or A_μ if we use 4 dimensions, as in relativity where m usually ranges from 0 to 3 with 0 being the time component. Under these conditions, the scalar product between two vectors, **A** and **B**, is just $\sum_{i=1}^{3} A_i B_i \; or$ in even more concise notation using the Einstein summation convention $\mathbf{A}\cdot\mathbf{B} = A_i B_i$. Recall that repeated indices indicate a sum in the Einstein summation convention and are called dummy variables or indices. If we want to write the full vector notation, we need to incorporate the basis vectors so that $\mathbf{A} = A_i\mathbf{e}_i$. All vector operations previously defined are valid, only now we use index notation. It is helpful to use the Kronecker delta, $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. Another very useful device is the Levi-Civita or permutation symbol, ε_{ijk}. As you probably remember, ε_{ijk} is +1 for a cyclic permutation and -1 for a noncyclic permutation. Cyclic simply means that the order of ijk remains unchanged, whereas noncyclic means they do change. Any repeated index gives zero for ε_{ijk}. For now, we will consider these to be useful properties of these symbols, but later we will see that they are special tensors known as isotropic tensors. It is not hard to see that $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is represented by $C_i = \varepsilon_{ijk}A_jB_k$. The so-called ε – δ identity is a very useful one to have on hand. It is $\varepsilon_{ijk}\varepsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$. Suppose we use this index to prove the BAC – CAB identity. You will see that it is much easier then writing out all the components. It is also a useful exercise to practice keeping track of the indices.

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\times\mathbf{D} = \mathbf{E}, \text{so } \mathbf{B}\times\mathbf{C} = \varepsilon_{ijk}B_jC_k = D_i.$$

Now $\mathbf{A}\times\mathbf{D} = \varepsilon_{rsi}A_sD_i = \varepsilon_{rsi}A_s\varepsilon_{ijk}B_jC_k = E_r = \varepsilon_{rsi}\varepsilon_{ijk}A_sB_jC_k = \varepsilon_{irs}\varepsilon_{ijk}A_sB_jC_k$.

So $E_r = (\delta_{rj}\delta_{sk} - \delta_{rk}\delta_{sj})A_sB_jC_k = \delta_{rj}\delta_{sk}A_sB_jC_k - \delta_{rk}\delta_{sj}A_sB_jC_k = B_r(A_kC_k) - C_r(A_jB_j)$, which is $\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$. If you don't like to deal with the cyclic and noncyclic cases, just use $\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$.

NEXT TIME: Derivatives in index notation, rotations and orthogonal transformations, and curvilinear coordinate systems.