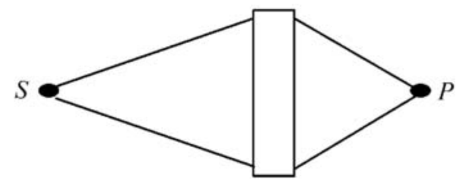
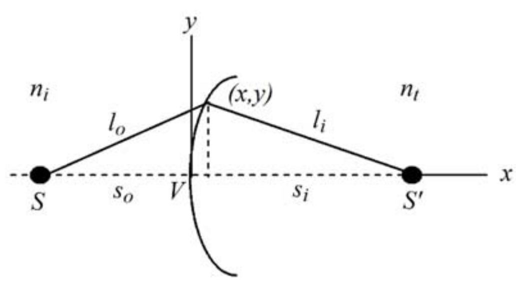
LAST TIME: Motivation for optics, Fermat's principle, Snell's law, law of reflection, and applications

Recall that flat surfaces serve as redirectors of light. Curved surfaces, however, can change diverging light rays to converging and vice versa, so they can form images. The figure to the right shows this effect. If *P* is an image of *S*, then our eye would not detect any difference between *S* and *P*, except maybe the size. A lens is a refracting device that reconfigures the energy. The easiest curved surface to make and to treat theoretically is the spherical surface, so we will concentrate mostly on these surfaces. Aspherical surfaces, such as ellipses and hyperbolas, are usually used to convert point sources into collimated beams or collimated beams into point sources. See your text for a brief discussion of these. Before we get too far along in our discussion of lenses, let me make some general comments that will be important as we develop our treatment of imaging. We have already seen that refraction is inherently a nonlinear process. Ultimately, this means that it is not possible to take a planar object and transform it into planar image using a nonlinear process. After all, planes are represented by linear relationships. To make reasonable progress in imaging with computational power, we will have to make approximations.



One interesting device is the Cartesian oval of revolution. Here is a figure that shows how it works. For *S′* to be an image of *S*, the optical paths along the optical axis and any other path must be equal. Therefore, $n_i s_o + n_t s_i = n_i l_o + n_t l_i$. l_o and l_i may also be written in terms of the arbitrary point (*x*,*y*) shown in the figure as



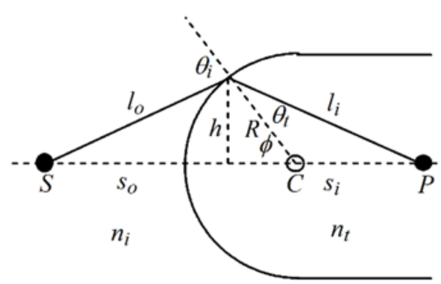
$$l_o = [(s_o + x)^2 + y^2]^{1/2}$$

and

$$l_i = [(s_i - x)^2 + y^2]^{1/2}.$$

If we substitute these values for l_o and l_i into the previous equation, we obtain an equation for the surface. The problem is that the shape of the surface depends on s_o and s_i. Furthermore, if s_o is not located on the optical axis, the shape of the surface is also changes. This situation is further evidence that our earlier notion that it is not possible to transform a plane into another plane with a nonlinear transformation.

Let's look now at Snell's law applied to a spherical surface. The figure to the right shows a ray originating at *S* and focusing at *P*. This figure is similar to the one above, but the surface here is specified to be spherical. The optical length *OPL* is given by



$$OPL = n_i l_o + n_t l_i.$$

Our objective here is to determine the variation of the optical path length as ϕ varies. We calculate the derivative with respect to ϕ, set it equal to zero, and see what Fermat's principle gives us. We use the law of cosines to determine l_o and l_i. Then the OPL is given by

$$OPL = n_i[R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi]^{1/2} + n_i[R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi]^{1/2}$$

We calculate $\frac{d(OPL)}{d\phi}$ and set it equal to zero to obtain

$$\frac{n_i}{l_o} + \frac{n_t}{l_i} = \frac{1}{R}\left(\frac{n_t s_i}{l_i} - \frac{n_i s_o}{l_o}\right).$$

Although this equation is exact, it is not easy to use because we must always know where the ray strikes the surface to use it to determine s_i if we know s_o. We make some headway if we let $\phi \ll 1$ so that $\cos\phi \cong 1$ and $s_o \cong l_o$ and $s_i \cong l_i$. Under these approximations, our equation simplifies to

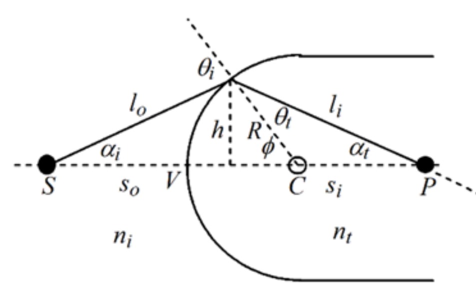
$$\frac{n_i}{s_o} + \frac{n_t}{s_i} = \frac{n_t - n_i}{R}.$$

The small angle approximation tells us that we could have started with the linearized version of Snell's law given by

$$n_i\theta_i \cong n_t\theta_t.$$

I will have you do this calculation as homework. This result is not particularly useful because it assumes light is incident onto a semi-infinite medium. We need to modify the equation so we may put a second spherical surface and make a lens. Before we do so, however, let's make some definitions concerning the equation above. When $s_i \to \infty, s_o = f_o$, so we have $f_o = \frac{n_i R}{n_t - n_i}$, where f_o is called the object focal length. Similarly, $s_o \to \infty, s_i = f_i$, so we have $f_i = \frac{n_t R}{n_t - n_i}$, where f_i is the image focal length.

Let's look again at our spherical surface and introduce some new angles that are more accessible than θ_i and θ_t. We call these angles α_i and α_t as shown in the figure. $\theta_i = \alpha_i + \phi$ and $\phi = \theta_t - \alpha_t$. Note that α_t is negative because it is below the horizontal. Therefore,



$$n_i(\alpha_i + \phi) = n_t(\alpha_t + \phi)$$

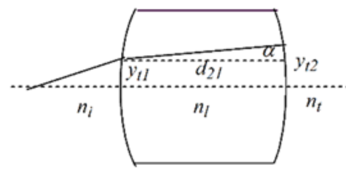
and

$$n_t\alpha_t = n_i\alpha_i - (n_t - n_i)\,\phi = n_i\alpha_i - (n_t - n_i)\frac{y_i}{R},$$

where we set $h = y_i$. Define $D = \frac{n_t - n_i}{R}$ = refractive power. Of course $y_t = y_i$. Refraction does not change the location of the ray with respect to the optical axis, but does change the angle. These results may be summarized neatly using matrices as follows.

$$\begin{bmatrix} n_t\alpha_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n_i\alpha_i \\ y_i \end{bmatrix}.$$

Define $\mathcal{R} = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix}$ as the refraction matrix and $\begin{bmatrix} n\alpha \\ y \end{bmatrix}$ as the ray vector. $\mathcal{R}$ tells us what happens as as ray is refracted at a surface. The next step in making a lens is to transfer or translate the ray to the next surface. The figure to the right shows this step. There is no refraction, so $n_l\alpha$ remains constant. However, $y_{t2} = y_{i1} + \alpha d_{21}$. To give these results, the matrix form is given by



$$\begin{bmatrix} n_l\alpha \\ y_{t2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{d_{21}}{n_l} & 1 \end{bmatrix} \begin{bmatrix} n_l\alpha \\ y_{t1} \end{bmatrix}.$$

We define the transfer matrix as

$$\mathcal{T} = \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix}.$$

This ray will undergo another refraction at the second surface with the refraction matrix given by

$$\mathcal{R}_2 = \begin{bmatrix} 1 & -D_2 \\ 0 & 1 \end{bmatrix} \text{ with } D_2 = \frac{n_t - n_l}{R_2}.$$

Therefore, to refract a ray at the first surface, transfer it to the second surface, and refract it at the second surface requires us to multiply three matrices with the result given by $\mathcal{R}_2\mathcal{T}_{21}\mathcal{R}_1$. This matrix is called the system matrix, usually designated by $\mathcal{A}_{21}$. Because matrix multiplication is not commutative, the matrices are thought of as progressing from right to left instead of left to right. Before we begin to use these matrices to obtain useful results, let's quickly review matrix multiplication. Consider matrix $\overleftrightarrow{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and matrix $\overleftrightarrow{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. The matrix multiplication is given by

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

The mulitplication may also be written in subscripted notation as $C_{ij} = A_{ik}\, B_{kj}$. The Einstein summation convention is understood; that is, repeated indices are summed. Written in this form, it is exactly like the matrix multiplication. However, notice that it is also acceptable to write $C_{ij} = B_{kj}\, A_{ik}$ because the elements of the matrices do commute. This is not, however, matrix multiplication. Now, $C_{11} = B_{11}A_{11}+B_{21}A_{12}$, $C_{12} = B_{12}A_{11}+B_{22}A_{12}$, $C_{21} = B_{11}A_{21}+B_{21}A_{22}$, and $C_{22} = B_{12}A_{21}+B_{22}A_{22}$. This method is much harder to keep track of the summation because it does **NOT** follow traditional matrix multiplication.

Let's look at how this process works for thin lenses. For thin lenses, $d_{21} \cong 0$, so the system matrix is given by

$$\mathcal{A}_{21} = \mathcal{R}_2\, \mathcal{T}_{21}\, \mathcal{R}_1 = \begin{bmatrix} 1 & -D_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -D_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix} \text{ with } D = D_1 + D_2.$$

The system matrix only gets us from the first surface in the optical system to the last surface in the optical system. How do we determine the equation for creating an image of na object?

Suppose we place a point object a distance s_o from the first surface of the optical system. We wish to know where the image of that object is formed; i.e., what is s_i? In order to determine this, we must transfer the object to the first surface of the optical system and then transfer the output ray from the last surface to the final image location. In this process, we must also define what we mean by the image location. Here is the mathematics.

$$\begin{bmatrix} n_i\alpha_i \\ y_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{s_i}{n_i} & 1 \end{bmatrix} \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{s_o}{n_o} & 1 \end{bmatrix} \begin{bmatrix} n_o\alpha_o \\ y_o \end{bmatrix}.$$

Notice that the equations are best read from the right to the left. We have the object ray vector, then the transfer to the system, and finally the transfer to the image location. The result, using step-by-step matrix multiplication, is given by

$$\begin{bmatrix} n_i\alpha_i \\ y_i \end{bmatrix} = \begin{bmatrix} 1 & -D \\ \frac{s_i}{n_i} & -D\left(\frac{s_i}{n_i}\right)+1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{s_o}{n_o} & 1 \end{bmatrix} \begin{bmatrix} n_o\alpha_o \\ y_o \end{bmatrix}$$
$$= \begin{bmatrix} 1-D\left(\frac{s_o}{n_o}\right) & -D \\ \left(\frac{s_i}{n_i}\right)+\left(\frac{s_o}{n_o}\right)\left(1-D\left(\frac{s_i}{n_i}\right)\right) & 1-D\left(\frac{s_i}{n_i}\right) \end{bmatrix} \begin{bmatrix} n_o\alpha_o \\ y_o \end{bmatrix}.$$

Let's call this large matrix the object-image matrix because it does relate the object and image distances, so

$$\mathcal{O} = \begin{bmatrix} 1-D\left(\frac{s_o}{n_o}\right) & -D \\ \left(\frac{s_i}{n_i}\right)+\left(\frac{s_o}{n_o}\right)\left(1-D\left(\frac{s_i}{n_i}\right)\right) & 1-D\left(\frac{s_i}{n_i}\right) \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{bmatrix}.$$

Let's connect the object ray and image ray by the object-image matrix in generic terms to see how we might define the image location.

$$\begin{bmatrix} n_i\alpha_i \\ y_i \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{bmatrix} \begin{bmatrix} n_o\alpha_o \\ y_o \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{11}n_o\alpha_o + \mathcal{O}_{12}y_o \\ \mathcal{O}_{21}n_o\alpha_o + \mathcal{O}_{22}y_o \end{bmatrix}$$

Finally,

$$n_i\alpha_i = \mathcal{O}_{11}n_o\alpha_o + \mathcal{O}_{12}y_o \text{ and } y_i = \mathcal{O}_{21}n_o\alpha_o + \mathcal{O}_{22}y_o.$$

If something is an image, then the height of the image y_i cannot depend on the angle α_o at which the object ray left; *i. e.*, $y_i \neq y_i(\alpha_o)$. This means that the imaging condition is given by $\mathcal{O}_{21} = 0$!! For our specific case here, then, we have

$$\left(\frac{s_i}{n_i}\right)+\left(\frac{s_o}{n_o}\right)\left(1-D\left(\frac{s_i}{n_i}\right)\right) = 0$$

Let's keep the lens in the same environment (air) on both sides so that $n_o = n_i = n_m$. Then we obtain

$$\left(\frac{s_i}{n_m}\right) + \left(\frac{s_o}{n_m}\right) = D\left(\frac{s_o}{n_m}\right)\left(\frac{s_i}{n_m}\right).$$

Set $n_m = 1$ and divide by $s_i s_o$ to obtain

$$\frac{1}{s_o} + \frac{1}{s_i} = D = \frac{n_l - 1}{R_1} + \frac{1 - n_l}{R_2} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{f_o} = \frac{1}{f_i} = \frac{1}{f}.$$

This gives us the thin lens equation and the lens-maker's equation all at once. This equation will be easily modified when we wish to obtain similar equations for systems of lenses that may be thick.

NEXT TIME: Sign conventions, applications of the thins lens and lens-maker's equations, and types of lenses