Physics 6303 Lecture 1 August 20, 2018

Go over syllabus.

Write down some of the fundamental equations of physics.

$$m\ddot{\mathbf{r}} = \mathbf{F}; \quad \mathbf{L} = \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \quad \text{or} \quad L_i = I_{ij}\omega_j; \quad \boldsymbol{\Gamma} = \frac{\mathrm{d}\boldsymbol{L}}{dt}; \quad \mathbf{F} = -\boldsymbol{\nabla} U$$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho; \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0; \quad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \quad \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 \varphi = 0; \qquad \nabla^2 \varphi = -\frac{\rho}{\epsilon_o}$$

$$\frac{\partial J^\mu}{\partial x^\mu} = 0; \quad \frac{\partial F^{\mu\nu}}{\partial x^\mu} = \mu_o J^\mu; \quad \frac{\partial G^{\mu\nu}}{\partial x^\mu} = 0$$

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) + V(\mathbf{r}, t)$$

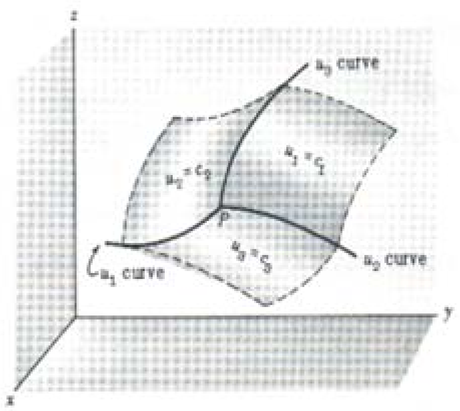
Other equations include the Klein-Gordon equation, the Lippmann-Schwinger equation, and the Boltzmann transport equation from statistical physics.

What are some of the common features of these equations? What determines the particular coordinate system in which we work?

As you know, vectors do not change as we change coordinate systems, but their components do change. Similarly, the operators such as $\boldsymbol{\nabla}, \boldsymbol{\nabla}\cdot, \boldsymbol{\nabla}\times,$ and ∇^2 change form in different coordinate systems. Clearly, we must know how to deal with multiple types of coordinate systems.

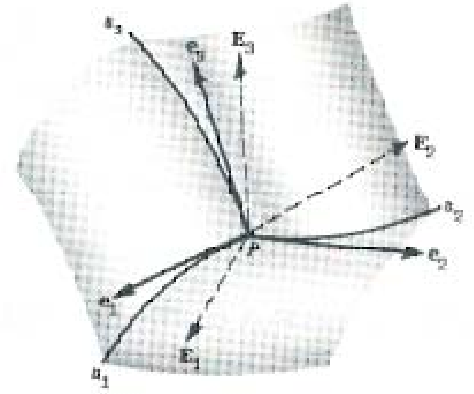
A good question to pose is “How are coordinate systems created?” You may never have thought much about this, but it is a very important question, so let’s look further into it.

In the most general sense, coordinate systems are constructed by intersecting surfaces. On the following page is a figure that shows three such arbitrary surfaces intersecting.



Arbitrary intersecting surfaces

These are surfaces defined by $u_1 = c_1$, $u_2 = c_2$, and $u_3 = c_3$. The intersections of the surfaces are defined by the u_1 curve, the u_2 curve, and the u_3 curve as shown. For the rectangular Cartesian coordinate system, what are these surfaces? What are they for the cylindrical coordinate system and for the spherical coordinate system? Do these surfaces have to intersect with the resulting curves being orthogonal to one another? The graph below this one shows the coordinate systems that are created by these intersections in general.



Two coordinate systems created
by intersecting surfaces

Do you see the difference between the coordinate system defined by the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ and $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$?

For consistency with what follows, let's change $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ to $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$.

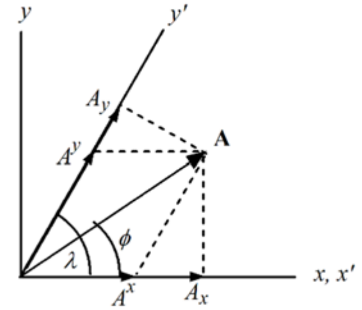
Then, a vector $\mathbf{A}$ may be written as

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3.$$

$$A^i = \text{contravariant components}$$
$$A_i = \text{covariant components}$$

The subscripted and superscripted basis vectors are dual or reciprocal basis vectors. They need not be unit vectors.

Armed with this information, do you see what makes the rectangular Cartesian coordinate system so nice to deal with?



A simple two-dimensional example serves to illustrate these ideas. Consider the figure to the left.
When you are asked to construct the components of vector $\mathbf{A}$, you have two choices as to how to do this. Do you see what they are?

You might notice an additional problem well. The contravariant components (superscripted variables) nicely form a normal vector addition to get $\mathbf{A}$. The covariant components (subscripted variables), however, do not. Can you guess the reason for this?

We will return to this problem later.
We can characterize coordinate systems into three rather distinct cases.

1. Rectangular Cartesian systems – constant unit vectors
2. Orthogonal curvilinear coordinate systems – unit vectors vary with position making derivatives more challenging
3. Nonorthogonal (oblique) coordinate systems – must distinguish between covariant and contravariant components

Let's start by considering vectors in the rectangular Cartesian system. First, how do we define a vector?

We can give the nonmathematical operational definition, we can give the transformation properties, or we can give the mathematical operations that the vectors must satisfy (vector space).

The operational definition simply says that a vector is an object that has magnitude and direction. Recall that a scalar is just a number, distance, mass, temperature, *etc*.

We will see next time how a vector behaves under rotations of a coordinate system. A word about notation is in order at this time. Arfken uses unit vectors given by $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y,$ and $\hat{\mathbf{e}}_z$. However, I prefer just to use $\hat{\mathbf{x}}, \hat{\mathbf{y}},$ and $\hat{\mathbf{z}}$. This notation allows us to use carets (hats) for all of the common systems by just using the appropriate coordinate with a hat over it and avoiding many subscripts.

Review some vector algebra and vector calculus.

If two vectors $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\,\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$ and $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$, the following statements are true.

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + \left(A_y + B_y\right)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}},$$

$$\mathbf{A} - \mathbf{B} = (A_x - B_x)\hat{\mathbf{x}} + \left(A_y - B_y\right)\hat{\mathbf{y}} + (A_z - B_z)\hat{\mathbf{z}},$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \text{ commutative law} - \text{addition},$$

$$\mathrm{c}(\mathbf{A} + \mathbf{B}) = \mathrm{c}\mathbf{A} + \mathrm{c}\mathbf{B} \text{ distributive law} - \text{addition},$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \text{ associative law} - \text{addition}$$

There are two ways of multiplying vectors: scalar or dot product, and vector or cross product:

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

and

$$\mathbf{A} \times \mathbf{B} = AB\sin\theta\,\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane formed by **A** and **B**. Recall that θ is the smallest angle between **A** and **B** and the right hand is used to determine the direction of $\hat{\mathbf{n}}$. In the coordinate-system dependent definition and in rectangular Cartesian coordinates, the cross product is given by

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\mathbf{x}}\left(A_y B_z - B_y A_z\right) - \hat{\mathbf{y}}\left(A_x B_z - B_x A_z\right) + \hat{\mathbf{z}}\left(A_x B_y - B_x A_y\right).$$

Vector Review Example

Consider two vectors given by **A** = $5\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$ and **B** = $-2\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$. (a) Find the magnitude of each and the angle each vector makes with the positive x axis. (b) Find the angle between the two vectors. (c) Find **A** · **B** and **A** × **B**.

Solution: (a) $A = [5^2 + 3^2]^{(1/2)} = \sqrt{34} = 5.83$ $\theta_A = \tan^{-1}(3/5) = 31.0^\circ$ (comment)

$B = [(-2)^2 + 4^2]^{(1/2)} = \sqrt{20} = 4.47$ $\theta_B = \tan^{-1}(4/-2) = -63.4^\circ$ (comment)

Instead $\theta_B = -63.4 + 180 = 117^\circ$

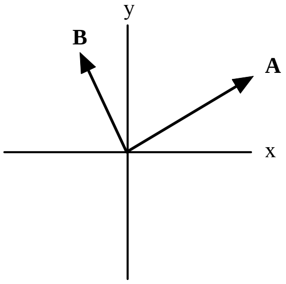
(b) Use $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y = A\,B \cos\theta$ to get

$$\cos\theta = \frac{A_x B_x + A_y B_y}{AB}$$

and

$$\cos\theta = \frac{(5)(-2) + (3)(4)}{(5.83)(4.47)},$$

so $\cos\theta = 0.0767$ and $\theta = 85.6^\circ$. We check to make sure this result makes sense by drawing the vectors.



So the figure looks reasonable and matches our calculation.

(c) $\mathbf{A} \cdot \mathbf{B} = 2$ as we see from the previous calculation and $\mathbf{A} \times \mathbf{B} = (5.83)(4.47)\sin(85.6^\circ)\,\hat{\mathbf{z}} = 26\hat{\mathbf{z}}$

Vector differentiation with respect to time in the rectangular Cartesian system is straightforward because the unit vectors are constant in direction. Remember that this is not true in the curvilinear coordinate systems (cylindrical and spherical).

Suppose we have a position vector $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}.$ In the most general case,

$$\mathbf{v}(t) = \hat{\mathbf{x}}\frac{dx(t)}{dt} + \hat{\mathbf{y}}\frac{dy(t)}{dt} + \hat{\mathbf{z}}\frac{dz(t)}{dt} + x(t)\frac{d\hat{\mathbf{x}}}{dt} + y(t)\frac{d\hat{\mathbf{y}}}{dt} + z(t)\frac{d\hat{\mathbf{z}}}{dt}.$$

As long as the coordinate system in not rotating, then all time derivatives of the unit vectors are zero. Then, as usual,

$$\mathbf{v}(t) = \hat{\mathbf{x}}\frac{dx(t)}{dt} + \hat{\mathbf{y}}\frac{dy(t)}{dt} + \hat{\mathbf{z}}\frac{dz(t)}{dt}.$$

The same holds true for the acceleration so

$$\mathbf{a}(t) = \hat{\mathbf{x}}\frac{dv_x(t)}{dt} + \hat{\mathbf{y}}\frac{dv_y(t)}{dt} + \hat{\mathbf{z}}\frac{dv_z(t)}{dt}.$$

It is useful to develop the spatial derivatives of vectors by considering the work done by a force.

Recall $W = \int_i^f \mathbf{F} \cdot d\mathbf{r}.$ In general, this line integral is path dependent. But if the integral is an exact differential, say $-dU$, then the work would be given by $W = -\Delta U$. Remember, however, that

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = \boldsymbol{\nabla} U \cdot d\mathbf{r}.$$

This motivates the definition of the gradient operator as $\boldsymbol{\nabla} = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$ in rectangular Cartesian coordinates. Because the gradient operator is a vector operator, it can occur as $\boldsymbol{\nabla} \cdot$ or $\boldsymbol{\nabla} \times$ which you should recognize as the divergence and the curl operator, respectively. It can also be dotted into itself to give the Laplacian operator ∇^2.

NEXT TIME: Examples of the operators, index formulation of vectors, Dirac delta functions, Levi-Civita symbol, rotations and orthogonal transformations, and curvilinear coordinate systems.