PHYS 6303 Homework 7 Solutions

1. (a) $m\ddot{y} = -T(\partial y(L)/\partial x) = 0 \quad (m=0)$

$\Rightarrow \dfrac{\partial y(L)}{\partial x} = 0$

(b) $y(x,t) = X(x)T(t)$ with $\dfrac{\partial^2 y(x,t)}{\partial t^2} = v^2 \dfrac{\partial^2 y(x,t)}{\partial x^2}$

$\therefore \dfrac{T''(t)}{T(t)} = v^2 \dfrac{X''(x)}{X(x)} = -\omega^2$ with $k^2 = \omega^2/v^2$

So $X(x) = A\sin kx + B\cos kx$

$X(0) = 0 \Rightarrow A(0) + B = 0 \Rightarrow B = 0$

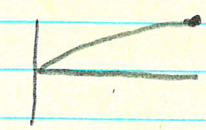
$X(x) = A\sin kx$ and $X'(x) = Ak\cos kx$

$Ak\cos kL = 0 \qquad k_n L = (2n+1)\pi/2$

$k_n = (2n+1)\pi/2L$

To better understand the physics, use $\lambda_n = \dfrac{2\pi}{k_n}$

$\lambda_n = \dfrac{2\pi(2)L}{(2n+1)\pi} = \dfrac{4L}{(2n+1)} \Rightarrow n=0 \quad L = \dfrac{\lambda_1}{4}$



$n=1 \quad \lambda_1 = \dfrac{4L}{3} \qquad L = \dfrac{3\lambda_1}{4}$ etc

$T = A\cos(\omega t) + B\sin(\omega t) \qquad \omega_n = k_n v$

$A + B$ undetermined without additional information.

2. This problem follows the same general pattern as the class example with two differences.
(1) If the bottom is in the x-y plane, you should use $\sinh k(L-z)$ so $V_T = 0$ at $L = z$. If you put the top in the x-y plane, $\sinh kz$ works.
(2) Because $V_0 =$ const, there can be no ϕ dependence so $m = 0$.

3. Use the expression for δ of a function

$$\delta[f(x)] = \sum_{x_i} \frac{1}{|f'(x_i)|}\,\delta(x-x_i) \qquad \text{with } f(x) = x^2+2x-3$$
$$= (x+3)(x-1)$$
$$f'(x) = 2x+2 = 2(x+1) \qquad x_1=-3,\ x_2=1$$

$$\delta(x^2+2x-3) = \frac{\delta(x+3)}{4} + \frac{\delta(x-1)}{4}$$

$$\int_{-\infty}^{\infty} e^{|x|}\,\delta(x^2+2x-3) = \frac{1}{4}\left(e^{3}+e^{1}\right)$$

4.(a) $\rho(\vec{r}) = \dfrac{M}{\pi R^2}\,\delta(z-0)$

(b) $$\vec{J} = I\,\delta(r-a)\,\delta(z)\,\hat{\phi} = I\,\delta(r-a)\,\delta(r\cos\theta)\,\hat{\phi}$$
$$= \hat{\phi}\, I\,\frac{\delta(r-a)}{r}\,\delta(\cos\theta) = I\,\delta(r-a)\,\frac{\delta(\theta-\pi/2)}{r\,|\sin\pi/2|}\,\hat{\phi}$$
$$= I\,\frac{\delta(r-a)}{r}\,\delta(\theta-\pi/2)\,\hat{\phi}$$

(c) $$I = \int I\,\frac{\delta(r-a)}{r}\,\delta(\theta-\pi/2)\, r\,dr\,d\theta$$
$$= I$$

5. $\dfrac{d^2y}{dx^2} = f(x) \qquad y(0)=0 \qquad \dfrac{dy}{dx}(1)=0$

$\dfrac{d^2G}{dx^2} = \delta(x-x')$ so $x \neq x'$ $\dfrac{d^2G}{dx^2} = 0$

$G(x,x') = A_1x + A_2 \quad x<x'$; $G(x,x') = B_1x + B_2 \quad x>x'$

$G(0,x')=0 \Rightarrow A_2 = 0 \quad x<x' \qquad G(x,x') = A_1x \quad x<x'$

$G'(x,x') = 0$ at $x=1 \Rightarrow B_1 = 0 \qquad G(x,x') = B_2 \quad x>x'$

Continuity of G at $x=x' \Rightarrow A_1x' = B_2$

Jump at $x=x' \Rightarrow G'\big|_{x-\epsilon}^{x+\epsilon} \Rightarrow 0 - A_1 = 1$

$\therefore \quad A_1 = -1 \qquad B_2 = A_1 x'$

$$G(x,x') = -x \qquad x < x'$$
$$= -x' \qquad x > x'$$

For $f(x) = x^2$, $y(x) = \int_0^x (-x')(x'^2)\,dx' + \int_x^1 (-x)\,x'^2\,dx'$

$$= -\frac{x^4}{4} - x\int_x^1 x'^2\,dx' = -\frac{x^4}{4} - x\left[\frac{x^3}{3}\right]_x^1 = -\frac{x^4}{4} - \frac{x}{3} + \frac{x^4}{3}$$

$$= \frac{x^4}{12} - \frac{x}{3}$$

6. $\dfrac{d^2V}{dx^2} = -\dfrac{\rho(x)}{\epsilon_0} \qquad V(0) = V(a) = 0$

$\dfrac{d^2G}{dx^2} = \delta(x-x') \qquad$ (4π is usually used here.)

$x \neq x' \Rightarrow \dfrac{d^2G}{dx^2} = 0 \Rightarrow G(x,x') = Ax + B \qquad x < x'$

$$G(x,x') = Cx + D \qquad x > x'$$

$x < x' \quad G(0) = 0 \Rightarrow B = 0 \quad G = Ax$

$x > x' \quad G(a) = 0 \Rightarrow Ca + D = 0 \Rightarrow D = -Ca$

$$G = C(x-a)$$

at $x = x' \qquad Ax' = C(x'-a) \Rightarrow C = \dfrac{Ax'}{x'-a}$

Now $G(x,x') = A(x')\,x \qquad x < x'$

$$= A(x')\,x'\,\frac{(x-a)}{(x'-a)}$$

Integrating G across $x = x' \Rightarrow$

$$\frac{A(x')x'}{(x'-a)} - A(x') = 1 \Rightarrow \frac{a\,A(x')}{x'-a} = 1 \Rightarrow A(x') = \frac{x'-a}{a}$$

$\therefore \quad G(x,x') = \dfrac{(x'-a)x}{a} \qquad x < x'$

$$= \frac{x'(x-a)}{a} \qquad x > x'$$

$$V = \int_0^x \sin\left(\frac{\pi x'}{a}\right) x'\frac{(x-a)}{a}\,dx' + \int_x^a \sin\left(\frac{\pi x'}{a}\right)\frac{x(x'-a)}{a}\,dx'$$

This integrates with mathematicas assistance to

$$-\left(\frac{a}{\pi}\right)^2 \sin\frac{\pi x}{a} \Rightarrow V(x) = \frac{1}{\epsilon_0}\left(\frac{a}{\pi}\right)^2 \sin\frac{\pi x}{a}$$

Check $V(0)=0$ $V(a)=0$