Physics 6303 Homework 6 Solutions

1. You know that one way of writing the solution to the ODE given by $\frac{d^2x}{dt^2} + \omega_o^2 x = 0$ is

$$x(t) = A \sin \omega_o t + B \cos \omega_o t.$$

Suppose, however, that you did not know anything about the elementary functions sine and cosine, and solve the equation using a series solution method. Show that your series matches the series representation for the sine and cosine functions.

There are no singularities here, so use

$$x(t) = \sum_{n=0}^{\infty} a_n t^n \qquad \dot{x}(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} t$$

$$\ddot{x}(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) t^n$$

Equating terms with like powers of t gives,
using $m\ddot{x} + kx = 0$

$$m[(n+2)(n+1)a_{n+2}] + k a_n = 0$$

$$a_{n+2} = -\frac{k}{m}\left[\frac{a_n}{(n+2)(n+1)}\right] = -\omega_0^2 \frac{a_n}{(n+2)(n+1)}$$

$$a_2 = -\omega_0^2 \frac{a_0}{2\cdot 1}\ ; \quad a_3 = -\omega_0^2 \frac{a_1}{3\cdot 2\cdot 1}\ ; \quad a_4 = -\omega_0^2 \frac{a_0}{4\cdot 3\cdot 2\cdot 1}$$

$$a_5 = -\omega_0^2 \frac{a_1}{5\cdot 4\cdot 3\cdot 2\cdot 1} \quad \text{etc.}$$

a_0 and a_1 are the arbitrary constant so

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

2. Two concentric spheres have their centers at the origin. The inner sphere has radius a and is held at zero potential. The outer sphere has radius b and is held at potential $V(\theta) = V_o \sin^2\theta$. (a) Express $V(\theta)$ in terms of the Legendre polynomials $P_0(\cos\theta)$ and $P_2(\cos\theta)$. (b) Find the potential between the spheres.

$$V_o \sin^2\theta = V_o (1-\cos^2\theta)$$

$$1-\cos^2\theta = C_1 P_0(\cos\theta) + C_2 P_2(\cos\theta)$$

$$= C_1 + C_2/2\,(3\cos^2\theta - 1)$$

$$= C_1 - C_2/2 + 3C_2/2\,\cos^2\theta$$

$\therefore\quad C_1 - C_2/2 = 1 \qquad 3C_2/2 = -1$

$C_1 = 2/3 \qquad C_2 = -2/3$

$$\therefore\quad V_o \sin^2\theta = (2V_o/3)\,[P_0(\cos\theta) - P_2(\cos\theta)]$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta)$$

$V(a,\theta) = 0$ and $V(b,\theta) = (2/3)V_o\,[P_0(\cos\theta) - P_2(\cos\theta)]$

apply BCs

$$V(a,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell a^\ell + B_\ell a^{-(\ell+1)}\right) P_\ell(\cos\theta) = 0$$

This connects A_ℓ and B_ℓ so

$$A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} = 0 \quad \text{and} \quad B_\ell = -A_\ell a^{2\ell+1}$$

$$\therefore\quad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \left(r^\ell - \frac{a^{2\ell+1}}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Now apply BC at $r=b$ along with orthogonality

$$\int_{-1}^{1} \sum_{\ell=0}^{\infty} A_\ell \left(b^\ell - \frac{a^{2\ell+1}}{b^{\ell+1}}\right) P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)$$

$$= \int_{-1}^{1} \frac{2}{3} V_o \,[P_0(\cos\theta) - P_2(\cos\theta)]\, P_{\ell'}(\cos\theta)\, d\theta$$

$\ell' = 0$ & $\ell' = 2$ are the only terms that contribute.

$$RHS = \frac{2V_0}{3}\frac{2}{2(0)+1} = \frac{4V_0}{3} \qquad l'=0$$

$$= \frac{2V_0}{3}\frac{2}{5} = \frac{4V_0}{15} \qquad l'=2$$

$$\therefore\ A_0\left(1-\frac{a}{b}\right)\frac{2}{1} = \frac{4V_0}{3} \quad \text{and} \quad A_0 = \frac{2V_0 b}{3(b-a)}$$

Similarly, $A_2\left(b^2-\frac{a^5}{b^3}\right)\left(\frac{2}{5}\right) = \frac{4V_0}{15}$

$$A_2 = \frac{2V_0 b^3}{3(b^5-a^5)} \quad \text{and problem is solved.}$$

3. A sphere having radius R is centered at the origin and has potential $V_o \sin^2\theta \cos 2\phi$ on its surface. (a) Show how to write the potential in terms of spherical harmonics. (b) Determine the potential inside and outside the sphere.

$$V(r<R,\theta,\phi) = \sum_{l,m} A_{l,m}\, r^l\, Y_{l,m}(\theta,\phi)$$

$$V(R,\theta,\phi) = V_0 \sin^2\theta \cos 2\phi$$
$$= \tfrac{1}{2} V_0 \sin^2\theta\,(e^{2i\phi}+e^{-2i\phi})$$

Note $Y_{2,\pm 2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{\pm 2i\phi}$

$$\sin^2\theta\,(e^{2i\phi}+e^{-2i\phi}) = \sqrt{\frac{32\pi}{15}}\,(Y_{2,2}+Y_{2,-2})$$

$$\therefore\ V(R,\theta,\phi) = \frac{1}{2}V_0\left(\frac{32\pi}{15}\right)^{1/2}(Y_{2,2}+Y_{2,-2})$$

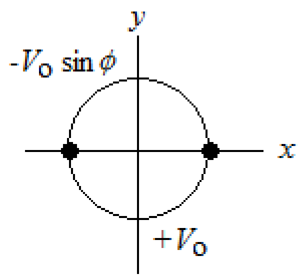
The orthogonality condition shows that only $l=2$, $m=\pm 2$ contribute.

$$\therefore\ A_{2,2} = A_{2,-2} = \frac{V_0}{R^2}\left(\frac{8\pi}{15}\right)^{1/2}$$

$$V(r<R,\theta,\phi) = \frac{V_0 r^2}{a^2}\left(\frac{8\pi}{15}\right)^{1/2}\left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta\cos 2\phi.$$

$$V(r<R,\theta,\phi) = \frac{V_0 r^2}{a^2}\left(\frac{8\pi}{15}\right)^{1/2}\left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta\cos 2\phi.$$

4. A long cylinder is cut in half so that the upper part ($0 < \phi < \pi$) can be held at a potential $-V_0 \sin\theta$, and the lower part ($\pi < \phi < 2\pi$) can be held at a potential $+V_0$ as shown in the figure. Let R be the radius of the cylinder. Write the most general solution to Laplace's equation for this geometry. (b) Assuming that you are to determine the potential outside the cylinder, state which constants in the general equation must be zero and why. (c) Determine the potential outside the cylinder.



$$V(\rho,\phi) = C_0 \ln\rho + D_0 + \sum_{m=1}^{\infty}\left(C_m\rho^{m} + D_m\rho^{-m}\right) \times \left(A_m\cos m\phi + B_m \sin m\phi\right)$$

Outside $C_0 \to 0$ $C_m \to 0$ to keep solution finite.

$$\therefore V(\rho,\phi) = D_0 + \sum D_m \rho^{-m}\left(A_m \cos m\phi + B_m \sin m\phi\right)$$

$$V(R,\phi) = -V_0 \sin\phi \qquad 0<\phi<\pi$$
$$= V_0 \qquad \pi<\phi<2\pi$$

$$\therefore V(R,\phi) = D_0 + \sum_{m=1}^{\infty} D_m R^{-m}\left(A_m\cos m\phi + B_m \sin m\phi\right)$$

Use orthogonality to get

$$D_m R^{-m} A_m = (1/\pi)\int_0^{2\pi} V(\phi)\cos m\phi\, d\phi$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}(-V_0 \sin\phi\cos m\phi\, d\phi + \int_{\pi}^{2\pi} V_0 \cos m\phi\, d\phi\right]$$

Simplify with trig identities and integrate

to get $D_m A_m = \frac{V_0 R^m}{\pi}\left(\frac{1+\cos m\pi}{m^2-1}\right)$

Now we have to repeat with

$$D_m R^{-m} B_m = \frac{1}{\pi}\int_0^{2\pi} V(\phi)\sin m\phi\, d\phi$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} -V_0 \underbrace{\sin\phi \sin m\phi\, d\phi}_{=0} + \int_{\pi}^{2\pi} V_0 \sin m\phi\, d\phi\right]$$

$$= \frac{1}{\pi}\frac{V_0}{m}\left(-\frac{\cos m\phi}{m}\right)_{\pi}^{2\pi} = \frac{V_0}{m\pi}\left(\cos m\pi - \cos 2m\pi\right)$$

$$= \frac{V_0}{m\pi}(\cos m\pi - 1)$$

$$\therefore D_m B_m = \frac{V_0}{m\pi R^{-m}}(\cos m\pi - 1)$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} V(\phi)\, d\phi = \frac{V_0}{2\pi}(\pi - 2)$$

where it was separated again.

All constants found.

5. Rodrigues' formula for Legendre polynomials is given by $P_\ell(\mu) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{d\mu^\ell}(\mu^2-1)^\ell$. Show that the first 3 Legendre polynomials, ℓ = 0 to 2, are correctly given by this formula.

$$P_\ell(\mu) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{d\mu^\ell}(\mu^2-1)^\ell$$

$$\ell = 0 \Rightarrow P_0(\mu) = \frac{1}{2^0 0!} = 1$$

$$l=1 \Rightarrow P_1(\mu) = \frac{1}{2^1 1!} \frac{d}{d\mu} (\mu^2 - 1) = \frac{1}{2} (2\mu) = \mu = \cos\theta$$

$$l=2 \Rightarrow P_2(\mu) = \frac{1}{2^2 2!} \frac{d}{d\mu} \left(\frac{d}{d\mu} (\mu^2-1)^2 \right) = \frac{1}{8} \frac{d}{d\mu} \left[2(\mu^2-1)(2\mu) \right]$$

$$= \frac{1}{8} \frac{d}{d\mu} \left(4\mu (\mu^2 - 1) \right) = \frac{1}{2} (3\mu^2 - 1)$$

$$= \frac{1}{2} \left(3\cos^2\theta - 1 \right)$$