Physics 6303 Homework 5 Solutions

1. Prove that the solutions to Laplace's equation are unique provided the potential is specified on the boundary.

The usual method is to assume that 2 solution V_1 and V_2 exist so that
$\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$ with V_1 and V_2 both satisfying the same BCs. Let $V_3 = V_1 - V_2$ with $\nabla^2 V_3 = 0$. $\therefore$ $V_3 = 0$ on the boundary since $V_1 = V_2$ on the boundary. Laplace's equation has no local maxima or minima in the region.
$\therefore$ $V_3 = 0$ everywhere $\Rightarrow$ $V_1 = V_2$ (unique)

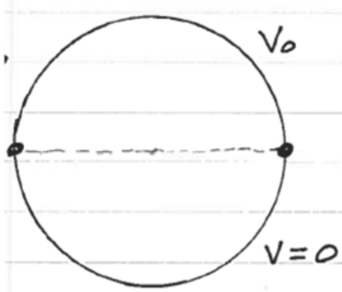
2. Prove that the eigenvalues of the Sturm-Liouville differential equation are real.

For this proof, use the same technique I used to prove orthogonality, except instead of using y_n and y_m, use y and y^*. The final result give

$$(\lambda - \lambda^*) \int_a^b w(x) \, |y|^2 dx = 0$$

$$\Rightarrow \lambda = \lambda^*$$

3. Reconsider the problem of the hollow conducting sphere of radius *R* that is divided into two halves, with the top half held at potential V_o and the bottom half grounded. Determine the potential outside the sphere. (HINT: We solved the problem inside the sphere in class.)



Find V outside $\Rightarrow A_\ell = 0$

so $V_{out} = \sum_{\ell=0} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

$$V(R,\theta) = V_o \quad \text{for } 0 < \theta < \pi/2$$
$$= 0 \quad \text{for } \pi/2 < \theta < \pi$$

Apply orthogonality

$$\int_{-1}^{1} \sum_{\ell=0}^{\infty} B_\ell R^{-(\ell+1)} P_\ell(\cos\theta) P_m(\cos\theta) \, d(\cos\theta)$$

$$= \int_0^1 V_o P_m(\cos\theta) \, d(\cos\theta) = B_m R^{-(m+1)} \frac{2}{2m+1}$$

To integrate the RHS use

$$P_m(\cos\theta) = \frac{1}{2m+1}\left[P'_{m+1}(\cos\theta) - P'_{m-1}(\cos\theta)\right]$$

$$\therefore \int_0^1 V_0 P_m(\cos\theta)\, d(\cos\theta) = \frac{1}{2m+1}\int_0^1 V_0\left[P'_{m+1}(\cos\theta) - P'_{m-1}(\cos\theta)\right] d(\cos\theta)$$

$$\therefore = \frac{1}{2m+1} V_0\left[P_{m+1}(1) - P_{m-1}(1) - P_{m+1}(0) + P_{m-1}(0)\right]$$

$$= -\frac{V_0}{2m+1}\left[P_{m+1}(0) - P_{m-1}(0)\right] \qquad P_m(1) = 1 \text{ for all } P_m$$

$$= B_m R^{-(m+1)} \frac{2}{2m+1}$$

$$\text{so } B_m = -\frac{V_0}{2} R^{(m+1)}\left[P_{m+1}(0) - P_{m-1}(0)\right]$$

and problem is solved.

4. Suppose a charge distribution is given by $\sigma_o \sin^2\theta$. (a) Determine the charge distribution in terms of Legendre polynomials. (b) What is the advantage of writing the charge distribution in this way?

$$\sigma(\theta) = \sigma_0 \sin^2\theta = \sigma_0(1-\cos^2\theta)$$

(a)

$$1 - \cos^2\theta = aP_0(\cos\theta) + bP_2(\cos\theta)$$

$$= a + b(1/2)(3\cos\theta^2 - 1)$$

$$= a + 3/2\, b\cos^2\theta - b/2$$

$$\therefore b = -2/3 \quad \text{and} \quad 1 = a - b/2$$

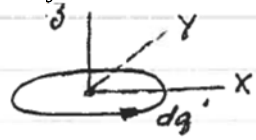
$$= a + 1/3$$

$$a = 2/3$$

$$\text{so } \sigma(\theta) = \sigma_0\left(\frac{2}{3}P_0 - \frac{2}{3}P_2\right)$$

(b) Having $\sigma(\theta)$ in terms of P_0 and P_2 makes it easy to use the orthogonality condition.

5. A circular ring having radius *R* and lying in the x-y plane with its center at the origin carries a uniformly distributed charge *q*. Calculate the electric potential everywhere for *r* > *R*.



$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq'}{|\vec{r}-\vec{r}\,'|} \qquad dq' = \frac{q}{2\pi R}\, R\, d\phi'$$

$$\vec{r} = z\hat{z} \qquad \vec{r}\,' = R\cos\phi'\,\hat{x} + R\sin\phi'\,\hat{y}$$

$$|\vec{r}-\vec{r}\,'| = |z\hat{z} - R\cos\phi'\,\hat{x} - R\sin\phi'\,\hat{y}|$$
$$= (z^2+R^2)^{1/2}$$

so $$V = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{q}{2\pi R}\frac{R\,d\phi'}{(z^2+R^2)^{1/2}} = \frac{2\pi}{4\pi\epsilon_0}\frac{q}{2\pi R}\frac{R}{(z^2+R^2)^{1/2}}$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{(z^2+R^2)^{1/2}}$$
$$= \frac{q}{4\pi\epsilon_0 z\,(1+R^2/z^2)^{1/2}} = \frac{q}{4\pi\epsilon_0 z}(1+R^2/z^2)^{-1/2}$$
$$= \frac{q}{4\pi\epsilon_0 z}\left(1 - \frac{R^2}{2z^2} + \frac{3}{8}\frac{R^4}{z^4} + \cdots\right)$$

For $r > R$ use $$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

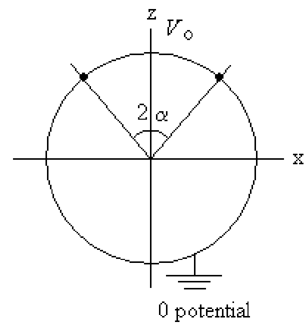
at $\theta = 0$ $r = z$, so we may match coefficients.

$$V(z,0) = \frac{B_0}{z}(1) + \frac{B_1}{z^2}(1) + \frac{B_2}{z^3}(1) + \frac{B_3}{z^4}(1)$$

$$\therefore\ B_0 = \frac{q}{4\pi\epsilon_0}; \quad B_1 = 0; \quad B_2 = -\frac{qR^2}{8\pi\epsilon_0}; \quad B_3 = 0; \text{ etc.}$$

Substituting these values of B into $V(r,\theta)$ gives the solution.

6. A conducting sphere having radius R is separated into two parts that are insulated from one another. The top portion is held at a constant potential V_o, whereas the bottom section is grounded (potential = 0). The angle that divides the top portion from the bottom section is 2α as shown. Calculate the potential everywhere for $r > R$.



(HS-7)

Outside $A_\ell \to 0$ so $V(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell \, r^{-(\ell+1)} P_\ell(\cos\theta)$

$$V(R,\theta) = V_0 \quad 0 < \theta < \alpha$$
$$= 0 \quad \alpha \le \theta \le \pi$$

Use orthogonality to obtain

$$\int_{-1}^{1} \sum_{\ell=0}^{\infty} B_\ell R^{-(\ell+1)} P_\ell(\cos\theta) P_m(\cos\theta)\, d(\cos\theta)$$
$$= \int_{1}^{\theta=\alpha} V_0 P_m(\cos\theta)\, d(\cos\theta)$$

$$\therefore \frac{B_\ell}{R^{\ell+1}} \frac{2}{2m+1} = V_0 \int_{1}^{\theta=\alpha} P_m(\cos\theta)\, d(\cos\theta)$$

Use $P_m(\mu) = \frac{1}{2m+1}\left(P'_{m+1}(\mu) - P'_{m-1}(\mu) \right)$

$$RHS = \frac{V_0}{2m+1} \int_{1}^{\cos\alpha} \left[P'_{m+1}(\cos\theta) - P'_{m-1}(\cos\theta) \right] d(\cos\theta)$$

$$= \frac{V_0}{2m+1} \left[P_{m+1}(\cos\theta) - P_{m-1}(\cos\theta) \right]_{1}^{\theta=\alpha}$$

$$= \frac{V_0}{2m+1} \left[P_{m+1}(\cos\alpha) - P_{m-1}(\cos\alpha) \right]$$ $P_\ell(1) = 1$ all P_ℓ's

$$\therefore\ B_\ell = V_0 R^{\ell+1} \frac{(2\ell+1)}{2(2\ell+1)} \left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha) \right]$$

$$= \frac{V_0 R^{\ell+1}}{2} \left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha) \right]$$

Put B_ℓ back into $V(r,\theta)$ for solution.