Physics 6303 Homework Set 4 Solution

1. If A^{ij} is symmetric, show that $A_{ij} = A_{ji}$.

$A^{ij} = A^{ji}$ is given

$$A_{kl} = g_{ki} g_{lj} A^{ij} = g_{ki} g_{lj} A^{ji} = g_{lj} g_{ki} A^{ji} = A_{lk}$$

2. The contravariant form of Newton's second law in a general coordinate system is given by

$$F^i = m\left(\ddot{x}^i + \Gamma^i_{kj}\dot{x}^k\dot{x}^j\right),$$

where Γ^i_{kj} is the Christoffel symbol of the second kind. (a) Use the metric for plane polar coordinates and its connection to the Christoffel symbols to write Newton's second law in plane polar coordinates. (b) Explain the physical meaning of the two components.

$$F^i = m(\ddot{x}^i + \Gamma^i_{kj}\dot{x}^k\dot{x}^j)$$

For plane polar coordinates, the metric is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad \Gamma^i_{kj} = \frac{1}{2} g^{li}\left[\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^l}\right]$$

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \qquad r = 1 \qquad \theta = 2$$

$$\Gamma^1_{11} = 1/2\, g^{11}\left[\partial g_{11}/\partial x^1 + \partial g_{11}/\partial x^1 - \partial g_{11}/\partial x^1\right]$$
$$+ 1/2\, g^{21}\left[\partial g_{12}/\partial x^1 + \partial g_{12}/\partial x^1 - \partial g_{11}/\partial x^2\right] = 0$$

All derivatives of g_{11}, g_{12}, and g_{21} wrt anything are zero.

The only term that can contribute is g_{22} and that will be the derivative wrt x_1.

$$\therefore\ \Gamma^1_{22} = (1/2)\, g^{11}\left[\partial g_{12}/\partial x_2 + \partial g_{12}/\partial x_2 - \partial g_{22}/\partial x^1\right]$$
$$= 1/2\ (1)\ (-2r) = -r$$

$$\Gamma^2_{12} = 1/2\, g^{22} \left[\partial g_{21}/\partial x_1 + \partial g_{22}/\partial x_1 - \partial g_{12}/\partial x_2 \right]$$
$$= 1/r = \Gamma^2_{21}$$

So $F^1 = m(\ddot{x}^1 + \Gamma^1_{22}\, \dot{x}^2 \dot{x}^2)$

$$F^r = m\ddot{r} - r\dot{\theta}^2$$

$$F^2 = m(\ddot{x}^2 + 2\Gamma^2_{12}\, \dot{x}^2 \dot{x}^1)$$

$$F^\theta = m(\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r}) \Rightarrow rF^\theta = m(\ddot{\theta} + 2\dot{r}\dot{\theta})$$

torque $= \frac{d}{dt}$ (ang mom)

generalized force $= \frac{d}{dt}$ (gen. mom.)

3. Show that the product of a matrix $A = \begin{bmatrix} a & b \\ \frac{-a^2}{b} & -a \end{bmatrix}$ with itself is zero. What conclusions do you draw from this observation?

$$A = \begin{bmatrix} a & b \\ -a^2/b & -a \end{bmatrix} \begin{bmatrix} a & b \\ -\frac{a^2}{b} & -a \end{bmatrix} = \begin{bmatrix} a^2 - a^2 & ab - ba \\ a^3/b + a^3/b & -a^2 + a^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note $\det A = 0$ so A is singular and cannot be inverted. It also means that a matrix product can $= 0$ without either matrix $= 0$.

4. Recall that we defined the cross product of two vectors by $(\mathbf{U} \times \mathbf{V})_i = \epsilon_{ijk} U_j V_k$. (a) Show that the tensor given by $T_{ij} = U_i V_j - V_i U_j$ can represent the components of the cross product. (b) Write the tensor in matrix form.

(a) $(\vec{U} \times \vec{V})_i = \epsilon_{ijk} U_j V_k \Rightarrow (\vec{U} \times \vec{V})_1 = \epsilon_{123} U_2 V_3 + \epsilon_{132} U_3 V_2$
$$= U_2 V_3 - U_3 V_2$$
$$(\vec{U} \times \vec{V})_2 = \epsilon_{231} U_3 V_1 + \epsilon_{213} U_1 V_3 = U_3 V_1 - U_1 V_3$$
$$(\vec{U} \times \vec{V})_3 = \epsilon_{312} U_1 V_2 + \epsilon_{321} U_2 V_1 = U_1 V_2 - U_2 V_1$$

(b) $T_{ij}' = U_i' V_j' - V_i' U_j' \Rightarrow T_{11} = 0\,;\ T_{22} = 0\,;\ T_{33} = 0$

$T_{12} = U_1 V_2 - V_1 U_2 = -T_{21}$

$T_{13} = U_1 V_3 - V_1 U_3 = -T_{31}$ Tensor is antisymmetric

$T_{23} = U_2 V_3 - V_2 U_3 = -T_{32}$

$$T_{ij}' = \begin{bmatrix} 0 & U_1V_2 - V_1U_2 & U_1V_3 - V_1U_3 \\ V_1U_2 - U_1V_2 & 0 & U_2V_3 - V_2U_3 \\ V_1U_3 - U_1V_3 & V_2U_3 - U_2V_3 & 0 \end{bmatrix}$$

5. Use the expressions for $A^i{}_j = \frac{\partial \bar{x}^i}{\partial x^j}$ and $B_i{}^j = \frac{\partial x^j}{\partial \bar{x}^i}$ to show that matrix $\mathbb{B}^T = \mathbb{A}^{-1}$ in general. Note that the index that appears first is the row index, and the second index is the column. This convention allows us to continue to use the same notation that we have used when all indices are up or down. Do not make this hard – it is not.

$B^T = A^{-1} \Rightarrow A B^T = I \ ;\ A^i{}_j = \dfrac{\partial \bar{x}^i}{\partial x^j} \ ;\ B_i{}^j = \dfrac{\partial x^j}{\partial \bar{x}^i},$

$\therefore \dfrac{\partial \bar{x}^i}{\partial x^j} \dfrac{\partial x^j}{\partial \bar{x}^k} = \delta^i_k$ as expected

6. In class, I showed you how to transform the electric field from a long, straight wire using the transformation for a contravariant electric field vector. Use the value I gave you for transforming the field as if it were a covariant vector to carry out the same calculation. Comment on your results.

Transform $B_i{}^j E_j = \bar{E}_j$

$$\therefore \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \left(\frac{\lambda}{2\pi\epsilon_0\rho^2} \right)$$

$$= \frac{\lambda}{2\pi\epsilon_0\rho^2} \begin{pmatrix} x\cos\phi + y\sin\phi \\ -\rho x\sin\phi + \rho y\cos\phi \\ 0 \end{pmatrix} = \frac{\lambda}{2\pi\epsilon_0\rho^2} \begin{bmatrix} \rho\cos^2\phi + \rho\sin^2\phi \\ -\rho^2\cos\phi\sin\phi + \rho^2\sin\phi\cos\phi \\ 0 \end{bmatrix}$$

$$= \frac{\lambda}{2\pi\epsilon_0\rho}$$

Covariant and contravariant components are not distinguishable in C.C.