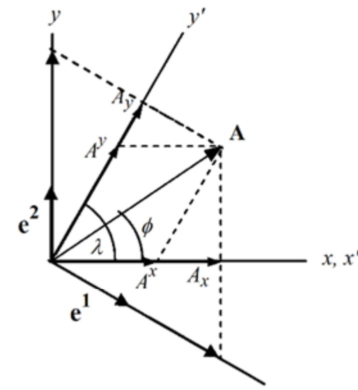
Physics 6303 Homework Set 3 Solution

1. We will use the nonorthogonal coordinate system one more time. Here is the figure I showed in class when we developed the dual basis vector system. Remember that I wrote the basis vectors for the contravariant components and the dual basis vectors for the covariant components. You will find these in my notes on page 3 of Lecture 4. (a) Calculate the metric tensor $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. (b) Calculate $g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j$. (c) Show that g^{ij} is the inverse of g_{ij}.



Use the results I gave in class for $\vec{e}_i$ and $\vec{e}^i$.
Recall $\vec{e}_1 = \hat{x}$, $\vec{e}_2 = \cos\lambda\,\hat{x} + \sin\lambda\,\hat{y}$, and $\vec{e}_3 = \hat{z}$ from which we obtain

$$\vec{e}^{\,1} = \frac{-\cos\lambda\,\hat{y} + \sin\lambda\,\hat{x}}{\sin\lambda}, \quad \vec{e}^{\,2} = \frac{\hat{y}}{\sin\lambda}, \text{ and } \vec{e}^{\,3} = \hat{z}$$

$\therefore g_{11} = \vec{e}_1 \cdot \vec{e}_1 = 1$, $g_{12} = \vec{e}_1 \cdot \vec{e}_2 = \cos\lambda$, $g_{21} = \vec{e}_2 \cdot \vec{e}_1 = \cos\lambda$,
and $g_{22} = \vec{e}_2 \cdot \vec{e}_2 = \cos^2\lambda + \sin^2\lambda = 1$.
$g_{13} = g_{23} = g_{31} = g_{32} = 0$ and $g_{33} = \vec{e}_3 \cdot \vec{e}_3 = 1$

$$\therefore g_{ij} = \begin{bmatrix} 1 & \cos\lambda & 0 \\ \cos\lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(symmetric as it must be)}$$

$$g^{11} = \vec{e}^{\,1} \cdot \vec{e}^{\,1} = \frac{\sin^2\lambda + \cos^2\lambda}{\sin^2\lambda} = \frac{1}{\sin^2\lambda}$$

$$g^{12} = \vec{e}^{\,1} \cdot \vec{e}^{\,2} = \frac{-\cos\lambda}{\sin^2\lambda} = g^{21}$$

$$g^{22} = \vec{e}^{\,2} \cdot \vec{e}^{\,2} = \frac{1}{\sin^2\lambda} \qquad g^{13} = g^{31} = g^{23} = g^{32} = 0 \quad g^{33} = 1$$

$$g_{ij}g^{ij} = \begin{bmatrix} 1 & \cos\lambda & 0 \\ \cos\lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\cos\lambda & 0 \\ -\cos\lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{1}{\sin^2\lambda}$$

$$= \begin{bmatrix} 1-\cos^2\lambda & 0 & 0 \\ 0 & -\cos^2\lambda+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{1}{\sin^2\lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

2. (a) From the figure, determine the covariant components of **A** using the angles given in the figure. (b) Using the expressions for g^{ij}, determine the values for the contravariant components of **A**. (c) Using the values determined here for A^i and A_i, obtain A_iA^i.

(a) $A_X = A\cos\phi \qquad A_Y = A\cos(\lambda-\phi)$

(b) $$A^X = g^{XX}A_X + g^{XY}A_Y = \frac{1}{\sin^2\lambda}A\cos\phi - \frac{1}{\sin^2\lambda}A\cos\lambda\cos(\lambda-\phi)$$

$$= \frac{1}{\sin^2\lambda}\left[A\cos\phi - A\cos\lambda\cos(\lambda-\phi)\right]$$

$$A^Y = g^{YX}A_X + g^{YY}A_Y = \frac{1}{\sin^2\lambda}\left[-A\cos\lambda\cos\phi + A\cos(\lambda-\phi)\right]$$

(c) $$A_iA^i = \frac{A\cos\phi}{\sin^2\lambda}\left[A\cos\phi - A\cos\lambda\cos(\lambda-\phi)\right]$$

$$+ \frac{A\cos(\lambda-\phi)}{\sin^2\lambda}\left[A\cos(\lambda-\phi) - A\cos\lambda\cos\phi\right]$$

$$= \frac{A^2}{\sin^2\lambda}\Big[\cos^2\phi - \cos\lambda(\cos\lambda\cos\phi + \sin\lambda\sin\phi)\cos\phi$$
$$+ (\cos\lambda\cos\phi + \sin\lambda\sin\phi)^2 - (\cos\lambda\cos\phi + \sin\lambda\sin\phi)$$
$$\times \cos\lambda\cos\phi\Big]$$

$$= \frac{A^2}{\sin^2\lambda}\Big[\cos^2\phi - \cos^2\lambda\cos^2\phi - \cos\lambda\sin\lambda\sin\phi\cos\phi$$

$$+ \cos^2\lambda\cos^2\phi + 2\cos\lambda\cos\phi\sin\lambda\sin\phi + \sin^2\lambda\sin^2\phi$$

$$- \cos^2\lambda\cos^2\phi - \sin\lambda\sin\phi\cos\lambda\cos\phi\Big]$$

$$= \frac{A^2}{\sin^2\lambda}\Big[\cos^2\phi - \cos^2\phi\cos^2\lambda + \cos^2\lambda\cos^2\phi + \sin^2\lambda\sin^2\phi - \cos^2\lambda\cos^2\phi\Big]$$

$$= \frac{A^2}{\sin^2\lambda}\left[\cos^2\phi + \sin^2\lambda\sin^2\phi - \cos^2\lambda\cos^2\phi\right] = \frac{A^2}{\sin^2\lambda}\left[\cos^2\phi(1-\cos^2\lambda) + \sin^2\lambda\sin^2\phi\right]$$

(con't) $$= \frac{A^2}{\sin^2\lambda}\left[\cos^2\phi\sin^2\lambda + \sin^2\phi\sin^2\lambda\right]$$

$$= \frac{A^2}{\sin^2\lambda}\left[\sin^2\lambda(\cos^2\phi + \sin^2\phi)\right] = A^2 \checkmark$$

3. Let the following set of components be defined in a two-dimensional Cartesian space:

$$A_{ij} = \begin{bmatrix} -y^2 & xy \\ xy & -x^2 \end{bmatrix}$$

(a) Does this set of coordinates transform as a tensor? (b) Explain. *Hint*: Try to represent this expression in terms of objects in index form that you already know are tensors. Consider the following set of components, again in two-dimensional Cartesian space:

$$B_{ij} = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$$

(c) Does this set of coordinates transform as a tensor? (d) Explain.

(a) $A_{ij} = \begin{bmatrix} -y^2 & xy \\ xy & -x^2 \end{bmatrix}$ in component form (2-dim)

$$\therefore A_{ij} = x_i x_j - r^2\delta_{ij} = x_i x_j - (x^2+y^2)\delta_{ij}$$

(b) $B'_{ij} = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$

$B_{ij} = x_i x_j \quad B_{11} = x^2 \quad B_{12} = xy \quad B_{21} = xy$

$B_{22} = y^2$

$x_i x_j$ = outer product (tensor)

In these forms, you can see that it is a tensor because each term has already been determined to be a tensor.

4. Recall that we wrote the electric quadrupole tensor as $Q_{ij} = \int (3\, x_i x_j - r^2 \delta_{ij}) \rho(\mathbf{x}) dV$. (a) From observation alone, how do you know this is a tensor? (b) Write Q_{xz} and Q_{zz}.

$Q_{ij} = \int (3 x_i x_j - r^2 \delta_{ij}) \rho(\vec{x}) dV$

(a) $x_i x_j$ = outer product $\quad \delta_{ij}$ = isotropic tensor

$\rho(x)$ + dV are scalars so Q_{ij} = tensor

(b) $Q_{xz} = \int (3xz\, \rho(x)\, dV$

$Q_{zz} = \int (3z^2 - (x^2+y^2+z^2)) \rho(\vec{x}) dV$

$= \int (2z^2 - x^2 - y^2) \rho(\vec{x}) dV$

5. (a) Determine the eigenvalues of the matrix given by

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

(b) Determine the normalized eigenvectors and the angle between them. (c) Why is the angle not 90 degrees?

(a) $\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = 0 \qquad (1-\lambda)(4-\lambda) - 6 = 0$

$\therefore \quad 4 - 5\lambda + \lambda^2 - 6 = 0 \Rightarrow \lambda^2 - 5\lambda - 2 = 0$

$\lambda = \dfrac{5 \pm [25 - 4(1)(-2)]^{1/2}}{2} = \dfrac{5 \pm 33^{1/2}}{2}$

$\lambda_1 = \frac{5+\sqrt{33}}{2} \cong 5.37 \qquad \lambda_2 = \frac{5-\sqrt{33}}{2} \cong -0.372$

(b) $\begin{pmatrix} 1-\lambda_1 & 2 \\ 3 & 4-\lambda_1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$

$\Rightarrow (1-\lambda_1)a_1 + 2a_2 = 0 \Rightarrow \quad 1-5.37a_1 + 2a_2 = 0$

$3a_1 + (4-\lambda_1)a_2 = 0 \Rightarrow \quad 3a_1 + (4-5.37)a_2 = 0$

check $3a_1 = 1.37a_2 \Rightarrow a_2 = 2.19a_1$

So $-4.37a_1 + 2a_2 = 0 \Rightarrow a_2 = \frac{4.37}{2}a_1 = 2.19a_1$

$a_1 = 1 \quad a_2 = 2.19 \quad (a_1^2 + a_2^2)^{1/2} = 2.40$

$a_1 = 0.416 \quad a_2 = 0.913$

$\begin{bmatrix} (1-\lambda_2) & 2 \\ 3 & 4-\lambda_2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0 \Rightarrow (1+0.372)b_1 + 2b_2 = 0$

$3b_1 + 4.372b_2 = 0$

$b_2 = -\frac{1.372}{2}b_1 = -0.686b_1$

$b_1 = 1 \quad b_2 = -0.686$

$(b_1^2 + b_2^2)^{1/2} = 1.21 \quad \therefore b_1 = 0.825 \quad b_2 = -0.567$

$\therefore \vec{a}\cdot\vec{b} = ab\cos\theta = a_1b_1 + a_2b_2$

$\cos\theta = (0.416)(0.825) + (0.913)(-0.567)$

$= -0.175$

$\theta \cong 100°$

Matrix is not symmetric, so the eigenvectors are not orthogonal.

6. For the triangular mass whose moment of inertia tensor I_{ij} I found in class, determine the principal moments of inertia and the principal axis coordinate system. See Lecture 5B.

$$I = \frac{m}{12}\begin{bmatrix} 6 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow \begin{vmatrix} 6-\lambda & -3 & 0 \\ -3 & 2-\lambda & 0 \\ 0 & 0 & 8-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)\left[(6-\lambda)(2-\lambda)-9\right]=0 \qquad (8-\lambda)(\lambda^2-8\lambda+3)=0$$

$$\lambda_3 = 8 \qquad \lambda_{1,2} = \frac{8 \pm [64-12]^{1/2}}{2} = 4 \pm \sqrt{13}$$

$$\lambda_1 = 7.61 \qquad \lambda_2 = 0.394$$ (I'm leaving out $m/12$ for now.)

λ_3:
$$\begin{bmatrix} 6-8 & -3 & 0 \\ -3 & 2-8 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -2\omega_x - 3\omega_y = 0 \\ -3\omega_x - 6\omega_y = 0 \\ \omega_3 \text{ arbitrary} \end{array}$$

$$\hat{e}_3 = (0, 0, 1)$$

λ_1:
$$\begin{bmatrix} 6-7.61 & -3 & 0 \\ -3 & 2-7.61 & 0 \\ 0 & 0 & 8-7.61 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -1.61\omega_x - 3\omega_y = 0 \\ -3\omega_x - 5.61\omega_y = 0 \\ 0.39\,\omega_3 = 0 \end{array}$$

$$\left.\begin{array}{l} \omega_x = -1.86\,\omega_y \\ \omega_x = -1.87\,\omega_y \end{array}\right\}$$ close enough (rounding) $\omega_3 = 0$

$$\omega_y = 1 \qquad \omega_x = -1.86$$

$$(\omega_y^2 + \omega_x^2)^{1/2} = 2.11$$

$$\omega_x = -0.881 \qquad \omega_y = 0.474$$

$$\hat{e}_1 = \begin{pmatrix} -0.881 \\ 0.474 \\ 0 \end{pmatrix} \qquad \hat{e}_2 = \begin{pmatrix} 0.471 \\ 0.885 \\ 0 \end{pmatrix}$$