Physics 6303 Homework Set 2 Solution

1. Consider the linear transformation of vector components given by $\begin{bmatrix} A'_x \\ A'_y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$. By the way, such 2-D transformations are called *affine transformations* and generally stretch, skew, reflect, or invert the objects being transformed. They are useful in computer graphics and modeling. (a) For what conditions on *a*, *b*, *c*, and *d* is this a distance preserving transformation? (b) Does the rotation matrix satisfy these conditions? (c) Calculate the inverse of the rotation matrix. (d) Calculate the determinant of the rotation matrix.

(a)

$$\begin{bmatrix} A_x' \\ A_y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} aA_x + bA_y \\ cA_x + dA_y \end{bmatrix}$$

$$A_x'^2 + A_y'^2 = a^2 A_x^2 + 2ab A_x A_y + b^2 A_y^2 + c^2 A_x^2 + 2cd A_x A_y + d^2 A_y^2$$
$$= (a^2 + c^2) A_x^2 + (b^2 + d^2) A_y^2 + (2ab + 2cd) A_x A_y$$

$$\therefore\ a^2 + c^2 = 1 \qquad b^2 + d^2 = 1 \qquad 2ab + 2cd = 0$$
$$ab = -cd$$

(b) $R = \begin{array}{c} \begin{array}{cc} a & b \\ \shortparallel & \shortparallel \end{array} \\ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ \begin{array}{cc} \shortparallel & \shortparallel \\ c & d \end{array} \end{array}$ in 2-D.

$$\therefore\ \cos^2\theta + (-\sin\theta)^2 = 1 \qquad \sin^2\theta + \cos^2\theta = 1$$
$$\cos\theta \sin\theta = -(-\sin\theta)\cos\theta = \cos\theta \sin\theta$$

(c) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

is the inverse of R.

$$\therefore\ A\cos\theta + C\sin\theta = 1 \qquad B\cos\theta + D\sin\theta = 0$$
$$-A\sin\theta + C\cos\theta = 0 \qquad -B\sin\theta + D\cos\theta = 1$$

$$B = -D\frac{\sin\theta}{\cos\theta} \qquad \therefore\ --D\frac{\sin^2\theta}{\cos\theta} + D\cos\theta = 1$$

$D \sin^2 + D \cos^2 = \cos\theta \Rightarrow D = \cos\theta$

$B\cos\theta + \cos\theta \sin\theta = 0 \Rightarrow \cos\theta\,(B + \sin\theta) = 0$

$B = -\sin\theta$

$A = \dfrac{C\cos\theta}{\sin\theta} \Rightarrow \dfrac{C\cos^2\theta}{\sin\theta} + C\sin\theta = 1$

$C(\cos^2\theta + \sin^2\theta) = \sin\theta \quad C = \sin\theta$

Finally, $-A\sin\theta + \sin\theta\cos\theta = 0$

$\sin\theta\,(-A + \cos\theta) = 0 \Rightarrow A = \cos\theta$

$\therefore R^{-1} = R^T$ by inspection.

(d) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2 + \sin^2\theta = +1.$

2. A particle moves in a circle of radius *R* at a constant speed *v*. Without writing components of vectors, (a) prove that the velocity is perpendicular to the position vector, (b) prove that the velocity is perpendicular to the acceleration, and (c) prove that the acceleration is oppositely directed to the position vector and has magnitude $\frac{v^2}{R}$.

(a) $\vec{r} \cdot \vec{r} = R^2 \Rightarrow \dfrac{d\vec{r}}{dt} \cdot \vec{r} + \vec{r} \cdot \dfrac{d\vec{r}}{dt} = 0 \Rightarrow 2\vec{r} \cdot \vec{v} = 0 \quad \vec{r} \perp \vec{v}$

(b) $\vec{v} \cdot \vec{v} = v^2 \quad \dfrac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \dfrac{d\vec{v}}{dt} = 0 \quad 2\vec{a} \cdot \vec{v} = 0 \quad \vec{a} \perp \vec{v}$

(c) $\vec{r} \cdot \vec{v} = 0 \Rightarrow \vec{v} \cdot \vec{v} + \vec{r} \cdot \vec{a} = 0$ so $\vec{r} \cdot \vec{a} = -v^2$

$\therefore$ $\vec{r}$ and $\vec{a}$ are anti-parallel $a = v^2/R$

3. (a) Calculate $\nabla \cdot (\mathbf{E} \times \mathbf{B})$ in terms of curl **E** and curl **B**. (b) What is the physical significance of the term $(\mathbf{E} \times \mathbf{B})$? You will use this identity frequently in electrodynamics.

$\vec{\nabla} \cdot (\vec{E} \times \vec{B})$ I'll do this in component form.

$$\vec{E} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & E_y & E_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(E_y B_z - B_y E_z) - \hat{y}(E_x B_z - B_x E_z) + \hat{z}(E_x B_y - B_x E_y)$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \frac{\partial}{\partial x}(E_y B_z - B_y E_z) - \frac{\partial}{\partial y}(E_x B_z - B_x E_z) + \frac{\partial}{\partial z}(E_x B_y - B_x E_y)$$

$$= \frac{\partial E_y}{\partial x} B_z + E_y \frac{\partial B_z}{\partial x} - \frac{\partial B_y}{\partial x} E_z - B_y \frac{\partial E_z}{\partial x}$$
$$- \frac{\partial E_x}{\partial y} B_z - E_x \frac{\partial B_z}{\partial y} + \frac{\partial B_x}{\partial y} E_z + B_x \frac{\partial E_z}{\partial y}$$
$$+ \frac{\partial E_x}{\partial z} B_y + E_x \frac{\partial B_y}{\partial z} - \frac{\partial B_x}{\partial z} E_y - B_x \frac{\partial E_y}{\partial z}$$

$$= \underbrace{B_x \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)}_{(\vec{\nabla} \times \vec{E})_x} + \underbrace{B_y \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right)}_{(\vec{\nabla} \times \vec{E})_y} + \underbrace{B_z \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)}_{(\vec{\nabla} \times \vec{E})_z}$$

$$+ \underbrace{E_x \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \right)}_{-(\vec{\nabla} \times \vec{B})_x} + \underbrace{E_y \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} \right)}_{-(\vec{\nabla} \times \vec{B})_y} + \underbrace{E_z \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right)}_{-(\vec{\nabla} \times \vec{B})_z}$$

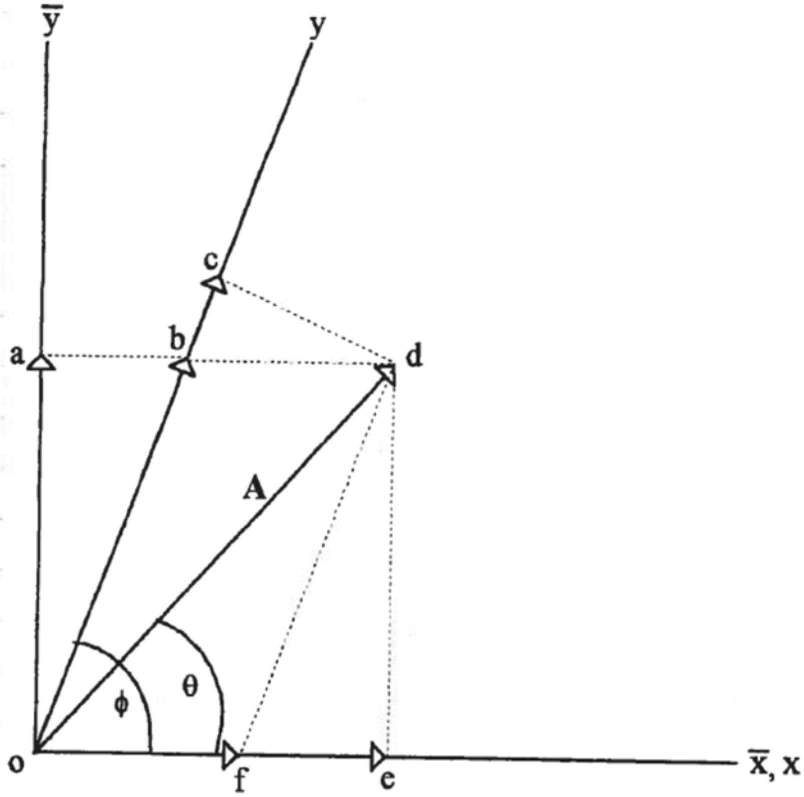
$$= \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}) \qquad QED$$

In index notation $\vec{\nabla} \cdot (\vec{E} \times \vec{B})_i = \frac{\partial}{\partial x_i} \epsilon_{ijk} E_j B_k$

$$= \epsilon_{ijk} \frac{\partial E_j}{\partial x_i} B_k + \epsilon_{ijk} \frac{\partial B_k}{\partial x_i} E_j$$

$$= \epsilon_{kij} \frac{\partial E_j}{\partial x_i} B_k - \epsilon_{jik} \frac{\partial B_k}{\partial x_i} E_j = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$$

4. Consider once more the nonorthogonal coordinate system with vector **A** represented as before. Using only algebra, geometry, and trigonometry, show that $A^2 = A_x A^x + A_y A^y$.



Then $\overline{oe} = A_{x'} = A^{x'} = A_x$; $\overline{of} = A^x$; $\overline{oa} = A_{y'} = A^{y'}$
$\overline{ob} = A^y$; $\overline{oc} = A_y$ To get the length of A,

$$(\overline{oc}^2 + \overline{cd}^2 + \overline{oe}^2 + \overline{de}^2) = 2A^2$$

$$\therefore A^2 = \tfrac{1}{2}(\overline{oc}^2 + \overline{cd}^2 + \overline{oe}^2 + \overline{de}^2)$$

$$\overline{de}^2 = \overline{df}^2 - \overline{ef}^2 = A^{y\,2} - (A_x - A^x)^2$$

$$\overline{cd}^2 = \overline{bd}^2 - \overline{bc}^2 = A^{x\,2} - (A_y - A^y)^2$$

$$\therefore A^2 = \frac{1}{2}\left(\hat{A}_y^{x\,2} + \hat{A}^{x\,2} - \hat{A}_y^{x\,2} + 2A_y A^y - \hat{A}^{y\,2}\right.$$
$$\left. + \hat{A}_x^{x\,2} + \hat{A}^{y\,2} - \hat{A}_x^{x\,2} + 2A_x A^x - \hat{A}^{x\,2}\right)$$
$$= 1/2\ (2A_x A^x + 2A_y A^y) = A_x A^x + A_y A^y$$

5. A solid object rotates with angular velocity **ω**. (a) Using cylindrical coordinates with the z-axis along the rotation axis, find the components of the velocity vector **v** at an arbitrary point within the body. (b) Use the expression for the curl in cylindrical coordinates to evaluate $\nabla \times \mathbf{v}$. Comment on your answer.

(a) Use $\vec{v} = \vec{\omega} \times \vec{\rho} = (0, \omega\rho, 0) \Rightarrow v_\phi = \omega\rho;\ v_\rho = 0;\ v_z = 0$

(b) Using the $\vec{\nabla}\times$ in cylindrical coordinates give only a z-component

$$\hat{z}\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial\phi}\right] = \hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}\omega\rho^2 = 2\omega\hat{z}$$

6. Following the procedure we used to derive the general expression for the divergence in curvilinear coordinates, derive the general expression for the curl in curvilinear coordinates.

$$\vec{\nabla}\times\vec{A} = \vec{\nabla}\times(A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3) = \vec{\nabla}\times(A_1\hat{e}_1) + \vec{\nabla}\times(A_2\hat{e}_2) + \vec{\nabla}\times(A_3\hat{e}_3)$$

$$= \frac{\hat{e}_2}{h_3h_1}\frac{\partial}{\partial u_3}(A_1h_1) - \frac{\hat{e}_3}{h_1h_2}\frac{\partial}{\partial u_2}(A_1h_1)$$

$$+ \frac{\hat{e}_3}{h_1h_2}\frac{\partial}{\partial u_1}(A_2h_2) - \frac{\hat{e}_1}{h_2h_3}\frac{\partial}{\partial u_3}(A_2h_2)$$

$$+ \frac{\hat{e}_1}{h_2h_3}\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\hat{e}_2}{h_3h_1}\frac{\partial}{\partial u_1}(A_3h_3)$$

In the first step of the problem, it is useful to remember that $\hat{\boldsymbol{e}}_1 = h_1\boldsymbol{\nabla}u_1$.

$$= \frac{\hat{e}_1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(A_3 h_3) - \frac{\partial}{\partial u_3}(A_2 h_2)\right] + \frac{\hat{e}_2}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(A_1 h_1) - \frac{\partial}{\partial u_1}(A_3 h_3)\right]$$

$$+ \frac{\hat{e}_3}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(A_2 h_2) - \frac{\partial}{\partial u_2}(A_1 h_1)\right]$$

$$\Rightarrow \vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

7. Consider the following matrix given by

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Does this matrix represent a rotation of the coordinate axes? If not, what transformation does it represent?

First of all, the determinant of the matrix is given by $-\cos^2\theta - \sin^2\theta = -1$. Therefore, the matrix does not represent a rotation. The easiest way I found to solve the problem was to use $\theta = \frac{\pi}{2}$ and see what happens to the axes. Doing this causes the x-axis and the y-axis to reflect around the line y = x tan(θ/2). We suspect, then, that this matrix is a reflection about some axis. Allow the general matrix to act on each of the unit vectors along x and y and you get the following results. $\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix}$ for the x and y axes, respectively. Putting in these axes allows you to see that both the x and y axis are reflected about the line y = x tan(θ/2). The figure below shows how this works. The angle α can be shown to be equal to the angle between y' and the dashed line, so the reflection around the line indicated above is correct.