Physics 6303 Homework Set 1 Solutions

1. In class, we did not consider the successive application of the scalar product and the vector product. (a) Show that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ may be written as a determinant with the components of each vector forming a row of the determinant. (b) What is the geometrical interpretation of this product? This product is the triple scalar product.

$\vec{A}\cdot\vec{B}\times\vec{C} = A_i(\vec{B}\times\vec{C})_i = A_i\,\epsilon_{ijk}B_jC_k = \epsilon_{ijk}A_iB_jC_k$

$\epsilon_{ijk}=0$ if any two indice are repeated so for $i=1$, we need sum only over 2 and 3.

$\epsilon_{1jk} = A_1(B_2C_3 + B_3C_2) = A_x(B_yC_z - C_yB_z)$

When written as a determinant

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = A_x(B_yC_z - B_zC) - A_y(B_xC_z - C_xB_z) + A_z(B_xC_y - C_xB_y)$$

Repeat ϵ_{2jk} and ϵ_{3jk} to get last 2 terms.

2. Suppose that a surface is defined by $\varphi(x,y,z)=c$, where c is a constant. Show that $\boldsymbol{\nabla}\varphi(x,y,z)$ is perpendicular to the surface.

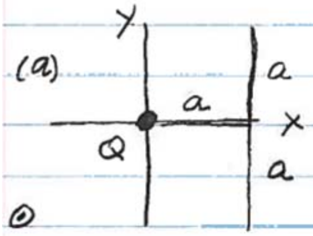
$\varphi(x,y,z)=c \Rightarrow d\varphi = 0 = \vec{\nabla}\varphi\cdot d\vec{r} = 0$

$\vec{r}$ is a vector from the origin to a point on the surface. $d\vec{r}$ lies in a plane tangent to the surface so $\vec{\nabla}\phi \perp d\vec{r}$ and is $\perp$ to the surface.

3. A point charge Q is at rest at the origin of a coordinate system. A square surface, $2a \times 2a$, in the y-z plane is located a distance a from the origin and is centered on the x-axis. (a) Calculate the electric field flux through the square surface by working out the surface integral. (b) Based on your result, state a far simpler way to determine the flux of the electric field through this surface.

(H1-2)

(a)



z out

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \qquad \vec{r}\,' = 0 \qquad \vec{r} = a\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{(a\hat{x} + y\hat{y} + z\hat{z})}{(a^2+y^2+z^2)^{3/2}}$$

$\Phi_E = \iint \vec{E}\cdot\hat{n}\, dS$ needs to be calculated with $\hat{n} = \hat{x}$ and $dS = dy\, dz$.

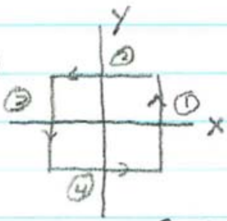
$$\therefore \Phi_E = \iint \frac{Q}{4\pi\epsilon_0}\frac{(a\hat{x} + y\hat{y} + z\hat{z})}{(a^2+y^2+z^2)^{3/2}}\cdot\hat{x}\, dy\, dz$$

$$= \frac{Qa}{4\pi\epsilon_0}\iint_{-a}^{a}\frac{dy\, dz}{(a^2+y^2+z^2)^{3/2}} = \frac{Qa}{4\pi\epsilon_0}\frac{2\pi}{3a} = \frac{Q}{6\epsilon_0}$$

(b) The symmetry of the problem allows you to surround the charge with a cube having side $2a$. $\therefore \oiint \vec{E}\cdot\hat{n}\, dS = Q/\epsilon_0$ and each surface has flux $Q/6\epsilon_0$

4. (a) Calculate the line integral $\oint \mathbf{F}\cdot d\boldsymbol{\ell}$, with $\mathbf{F} = x^3y^2(\hat{\mathbf{x}}+\hat{\mathbf{y}})$, around a square of side a centered at the origin in the x-y plane. (b) Calculate $\iint(\boldsymbol{\nabla}\times\mathbf{F})\cdot\mathbf{n}dS$ to verify Stokes' theorem for this case. (c) If $\mathbf{F}$ is a force, is it conservative?

(a)



$\vec{F} = x^3y^2(\hat{x}+\hat{y}) \qquad \int \vec{F}\cdot d\vec{r} = \int (F_x dx + F_y dy)$

Along ①: $dx = 0 \qquad x = a/2$ so $\vec{F} = (a^3/8)\,y^2(\hat{x}+\hat{y})$

$$\int \vec{F}\cdot d\vec{r} = \int_{-a/2}^{a/2} F_y\, dy = \int_{-a/2}^{a/2} (a^3/8)\, y^2 dy = (a^3/8)\, y^3/3 \Big|_{-a/2}^{a/2}$$

$$= (a^3/8)\, 2(a^3/8)/3 = a^3/96$$

Along ②: $dy = 0 \qquad \vec{F} = x^3\, a^2/4\, (\hat{x}+\hat{y})$

$$\int \vec{F}\cdot d\vec{r} = \int_{a/2}^{-a/2} F_x\, dx = \int_{a/2}^{-a/2} x^3\, a^2/4\, dx = a^2/4\, (x^4/4)\Big|_{a/2}^{-a/2} = 0$$

Along ③: $dx = 0 \qquad \vec{F} = (-a^3/8)\int_{a/2}^{-a/2} y^2 dy = (-a^3/8)\,(y^3/3)\Big|_{a/2}^{-a/2}$

$$= (-a^3/8)\, 2(-a^3/8)/3 = a^3/96$$

Along ④: $dy = 0$ but we are again integrating an odd function so $\int_{④} \vec{F}\cdot d\vec{r} = 0$

$$\oint \vec{F}\cdot d\vec{r} = a^3/48$$

(b) $\int (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dA \qquad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3y^2 & x^3y^2 & 0 \end{vmatrix} = \hat{x}(0) - \hat{y}(0) + \hat{z}(3x^2y^2 - 2x^3y)$

$\hat{n} = \hat{z} \qquad dA = dx\, dy$

$$\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} (3x^2y^2 - 2x^3y)\, dx\, dy = \int_{-a/2}^{a/2} (x^3y^2 - \overset{\to 0}{x^4/2\, y})\Big|_{-a/2}^{a/2} dy$$

$$= \int_{-a/2}^{a/2} 2\,(a^3/8)\, y^2\, dy = 2(a^3/8)\, y^3/3\Big|_{-a/2}^{a/2} = 2(a^3/8)(2/3)(a^3/8)$$

$$= a^3/48 \ ✓$$

(c) $\vec{\nabla}\times\vec{F} \neq 0 \qquad \oint \vec{F}\cdot d\vec{r} \neq 0 \qquad \vec{F}$ is not conservative

5. Consider the vector field $\mathbf{u} = k\mathbf{r}$, where $\mathbf{r}$ is the usual position vector in Cartesian coordinates. Verify that the divergence theorem holds for this field by calculating $\iiint (\boldsymbol{\nabla}\cdot\mathbf{u})dV$ and $\iint \mathbf{u}\cdot\mathbf{n}dS$, where S is the surface bounded by a cube of side a centered at the origin.

$\vec{u} = k\vec{r} \Rightarrow \vec{\nabla}\cdot\vec{u} = k\vec{\nabla}\cdot\vec{r} = k\vec{\nabla}\cdot(x\hat{x} + y\hat{y} + z\hat{z})$

$= k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3k$

$\therefore \iiint 3k\,dV = 3ka^3$

Now $\oiint \vec{u}\cdot\hat{n}\,dS = \oiint k(x\hat{x} + y\hat{y} + z\hat{z})\cdot\hat{n}\,dS$

On each face of the cube $\hat{n}$ is $\pm\hat{x}$, $\pm\hat{y}$, and $\pm\hat{z}$.

at $x = \frac{a}{2}$ $\displaystyle\int_{\substack{x=a/2 \\ \text{face}}} kx\hat{x}\cdot\hat{x}\,dx\,dy = \frac{ka}{2}a^2 = \frac{ka^3}{2}$

at $x = -a$

$\displaystyle\int_{\substack{x=-a/2 \\ \text{face}}} kx\hat{x}\cdot(-\hat{x})\,dx\,dy = k\left(-\frac{a}{2}\right)(-1)a^2$

$= \frac{ka^3}{2}$

Each face gives the same contribution.

$\therefore \quad 6\left(\frac{ka^3}{2}\right) = 3ka^3$ as required.