ECONOMICS 7330 – Probability and Statistics, Fall 2023

Homework 9. Due Wednesday October 25.

1. Let $X_1, ..., X_N$ be a sample of i.i.d. random variables with $\mathrm{E}\log(X_i) = 0$ and $\mathrm{Var}\log(X_i) = 1$. Let

$$Z_N = (X_1 * .... * X_N)^{\frac{1}{N}} .$$

Show that Z_n converges in probability to 1.
Let

$$Y_N = (X_1 * .... * X_N)^{\frac{1}{\sqrt{N}}} .$$

Show that Y_N converges in distribution to a log-normal distribution.

2. Let Y_N be a χ^2-distributed random variable with N degrees of freedom and let $Z_N = (Y_N - N)/\sqrt{2N}$. Show that Z_N converges in distribution to a $N(0,1)$ variable. (You can use without showing that the variance of a χ^2 distribution with k degrees of freedom is $2k$, although it follows easily from $E(x^2)^2 = 3$ and Ex^2=1 for a standard normal.)

3. Assume $\sqrt{n}(\hat{\theta} - \theta)$ converges in distribution to $N(0, \sigma^2)$.
Use the delta rule (aka delta method) to find the distribution of
a) $\hat{\theta}^4$.
b) $\frac{1}{1+2\,\hat{\theta}^2}$.

4. Assume that

$$\sqrt{n} \begin{pmatrix} \hat{\theta}_1 - \theta_1 \\ \hat{\theta}_2 - \theta_2 \end{pmatrix} \xrightarrow{d} N(0, \Sigma) .$$

What is the asymptotic distribution of
a) $2\hat{\theta}_1 + \hat{\theta}_2$.
b) $\exp(\hat{\theta}_1 + 2\hat{\theta}_2)$.

5. Assume that you have a sample of n observations from a Poisson distribution with probabilities $\pi(k) = \frac{\theta^k\, e^{-\theta}}{k!}$.
a) Write down the log-likelihood function $l_n(\theta)$.
b) Find the ML estimator $\hat{\theta}$.

5. Assume that you have a sample of n observations from a Pareto distribution with density $f(x) = \frac{\theta}{x^{1+\theta}}$.

a) Write down the log-likelihood function $l_n(\theta)$.
b) Find the ML estimator $\hat{\theta}$.