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ECONOMICS 7330 – Probability and Statistics, Fall 2025

Homework 8. Due Wednesday November 19.

1. Let X be distributed Poisson: $\pi(k) = \frac{\exp(-\theta)\theta^k}{k!}$ for nonnegative integer k and $\theta > 0$.
(a) Find the log-likelihood function $l_n(\theta)$.
(c) Find the asymptotic variance of $\hat{\theta}$. (That is σ^2 where $\sqrt(N)(\hat{\theta} - \theta)$ converges in distribution $N(0, \sigma^2)$.)

2. Assume that you have a sample of n observations from an exponential distribution with density $f(x) = \frac{1}{\theta} \exp^{-\frac{x}{\theta}}$. (The mean is θ and the variance is θ^2.)
a) Write down the log-likelihood function $l_n(\theta)$.
b) Find the asymptotic limit $l(\theta) = \lim_{n\to\infty} \frac{1}{n} l_n(\theta)$.
c) Show that the value of θ that maximizes $l(\theta)$ is equal to the "true" value that generated the data.

3. Consider the exponential model with density

$$f(t_i) = \frac{1}{\lambda_i} \exp(-t_i/\lambda_i)\,,$$

where λ_i is the mean (in a major application of the exponential distribution it is the expected waiting time for individual i). We assume we have a sample of N persons. We cannot estimate a parameter for each person, but we can estimate the effect of covariates by assuming $\lambda_i = x_i\beta$, where we now have a limited number of parameters as the dimension of β is the number of covariates ("regressors") included. (This is how we usually use the model.)
i) Find the Score and Hessian (or what Hansen calls "Likelihood Hessian" which is multiplied by –1). Note that when λ_i varies with i you need to find the expectation for each$i$ taking t_i as a random variable. The expression of course takes the same form for each i apart of the x_i that depends on i. Also assume the Law of Large numbers hold. (Note that the parameter is now β. In general, β is a vector, but you could start by solving the univariate case.)
ii) Use this to write down the asymptotic distribution for $N(\hat{\beta} - \beta)$.