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ECONOMICS 7330 – Probability and Statistics, Fall 2024

Homework 7. Due Wednesday October 16.

1. (Hansen Exercise 6.1) Assume a random sample $\{X_1, X_2, ..., X_n\}$ from a common distribution F with density f such that $E[X] = \mu$ and $var[X] = \sigma^2$ for a generic random variable $X \sim F$. The sample mean and variances are denoted $\overline{X}_n$ and $\hat{\sigma}^2 = \frac{1}{n}\Sigma_{i=1}^n (X_i - \overline{X}_n)^2$, with the bias-corrected variance $s_n^2 = \frac{1}{n-1}\Sigma_{i=1}^n (X_i - \overline{X}_n)^2$.

Suppose that another observation X_{n+1} becomes available. Show that
(a) $\overline{X}_{n+1} = (n\overline{X}_n + X_{n+1})/(n+1)$.
(b) $s_{n+1}^2 = [(n-1)s_n^2 + \frac{n}{n+1}(X_{n+1} - \overline{X}_n)^2]/n$
(This kind of updating is important in practice when n is very large and new observations regularly enter.)

2. (Hansen Exercise 6.11.) Let p be the unknown probability that a given basketball player makes a free throw attempt. The player takes n random free throws, of which he or she makes X of the attempts.
(a) Find an unbiased estimator $\hat{p}$ of p.
(b) Find $var(\hat{p})$.

3. (Hansen Exercise 6.12.) Suppose we know σ^2 and want our estimator to have a standard deviation smaller than a tolerance How large does n need to be to make this happen?

4. Let $X_1, ..., X_N$ be a sample of i.i.d. random variables with $\operatorname{E}\log(X_i) = 0$ and $\operatorname{Var}\log(X_i) = 1$. Let

$$Z_N = (X_1 * .... * X_N)^{\frac{1}{N}} .$$

Show that Z_n converges in probability to 1.
Let

$$Y_N = (X_1 * .... * X_N)^{\frac{1}{\sqrt{N}}} .$$

Show that Y_N converges in distribution to a log-normal distribution.

5. Let Y_N be a χ^2-distributed random variable with N degrees of freedom and let $Z_N = (Y_N - N)/\sqrt{2N}$. Show that Z_N converges in distribution to a $N(0,1)$ variable.

(You can use without showing that the variance of a χ^2 distribution with k degrees of freedom is $2k$, although it follows easily from $E(x^2)^2 = 3$ and Ex^2=1 for a standard normal.)

6. Assume $\sqrt{n}(\hat{\theta} - \theta)$ converges in distribution to $N(0, \Sigma)$,
with

$$\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Use the delta rule (aka delta method) to find the asymptotic variance of
a) $\hat{\theta}_1^3$.

b) $\frac{\hat{\theta}_2}{1+2\,\hat{\theta}_1^2}$.