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The Euler Equation-general form

Consider a consumer who maximize the *expected* utility of present and future consumption subject to some constraints. (Typically, a life time budget constraint but there might be other constraints involved, for example, no access to credit in some periods.) The criterion function is

$$U(c_t) + \frac{1}{1+\rho} EU(c_{t+1}) + (\frac{1}{1+\rho})^2 EU(c_{t+2}) + ...$$

where the time horizon may be finite or infinite (if the time horizon is infinite, it is implicit that the functions are such that the criterion function remain finite). Assume that the consumer has access to a financial asset (or several, there will be an Euler equation for each!) with a random (safe as a special case) return r_{t+1} (the return, like the earning on a stock is generally not known at period t). The consumer, however, knows the distribution of the return and takes that into account (rational expectations). The consumer is free to adjust *in period t and in period t+1* the amount invested in the asset by an amount x and adjust consumption correspondingly. Now assume that the consumer has solved the optimization problem and solved for the optimal consumption c_t and the have planned optimal future consumption $c_{t+1}, c_{t+2}, ...$ (When the consumer gets to period $t+1$ he or she will re-optimized based on what happens in between, for example, what the return on the consumers assets turned out to be.) At period t the consumer has the option of, for example, lowering consumption from its optimal level and invest it in the asset under consideration and add the gross returns to consumption in period $t+1$. This is feasible whatever the initial constraints on the problem are, as long as they do not constrain period t and $t+1$ actions.

So we can consider the feasible value of the criterion function

$$W = U(c_t - x) + \frac{1}{1+\rho} EU(c_{t+1} + x * (1 + r_{t+1})) + (\frac{1}{1+\rho})^2 EU(c_{t+2}) + ...$$

as a function of x. Of course, if the original consumption plans were optimal, that means that the derivative $\frac{dW}{dx}$ of W with respect to x is zero for $x = 0$. This gives the result that

$$0 \ = \ -U'(c_t) + E\{U'(c_{t+1}) \frac{1 + r_{t+1}}{1+\rho}\}$$

or

$$U'(c_t) \ = \ E\{U'(c_{t+1}) \frac{1 + r_{t+1}}{1+\rho}\} \ .$$

This is the Euler Equation, which has a very large number of applications in economics. (NOTE: be sure to observe that the expectation on the right-hand side involves the product of the marginal utility and the random return; making mistakes here is very costly at exam time.) If you look back at the way we solved the model with an infinite horizon and no uncertainty, the solution involved a) the Euler equation and b) the budget constraint. This is very typical.

Example. Assume that an agent can buy an amount S of a stock and an amount B of a bond. Either amount can be positive or negative. Assume the agent lives for 2 periods, period 1 and period 2. Exogenous income is Y_1 and Y_2, where Y_2 is random variable. Assume the interest on the bond, to be paid in period 2, is r_B and the net return to the stock is r_S, where r_S is a random variable. The agent now maximizes

$$U(Y_1 - B - S) + \frac{1}{1+\rho} EU(Y_2 + B * (1 + r_B) + S * (1 + r_S)) \, .$$

In this example, the agent chooses B and S, and this pins down next period's consumption as a random variable, with randomness in this example coming from income and the return to the stock. This is what is meant by "a consumption plan." Now if you take first-order conditions with respect to B and S, you get two Euler equations that you can solve for the unknown (optimal) values of B and S. For example, for the stock you get:

$$-U'(Y_1 - B - S) + \frac{1}{1+\rho} EU'(Y_2 + B * (1 + r_B) + S * (1 + r_S)) * (1 + r_S) = 0\,,$$

for the optimal values of B and S.

"Euler inquality" The interpretation of the Euler equation is that the utility value of the marginal dollar spent in period t ($U'(C_t)$ has to equal the marginal utility value of $(1 + r_{it})$ (what you get back from investing the dollar) in period $t + 1$ discounted by $\frac{1}{1+\rho}$. If an agent is constrained from "borrowing" in period t, the Euler equation would hold if the agent does not want to borrow (which one would check in fully specified model) but not if agent would want to borrow but cannot. (The reason I use borrowing in quotes is that the economy may not have a safe asset, and "borrow" may mean buying a negative amount of whatever asset the agent has access to.) An agent would like to borrow if the utility value of the marginal dollar exceeds the value of investing it (that is, $E\{U'(c_{t+1})\frac{1+r_{t+1}}{1+\rho}\}$). For an unconstrained agent, you might think of the agent as borrowing (moving consumption from $t + 1$ to t which lowers the marginal utility at t and increases it at $t + 1$ till equality holds) until he or she in indifferent between moving a dollar intertemporally. If the agent is constrained, he or she would have marginal utilities satisfying

$$U'(c_t) \;\geq\; E\{U'(c_{t+1})\frac{1 + r_{t+1}}{1+\rho}\}\ ,$$

with inequality if the constraint is binding.

Hall's PIH model

Hall assumed that agents optimized utility under uncertainty as in the previous paragraph, but further assumed that agents have access to a risk free asset with return r and that $r = \rho$. Hall further assumed that agents have a quadratic utility function

$$U(c_t) = c_t - \frac{a}{2}c_t^2$$

such that marginal utility is linear in consumption

$$U'(c_t) = 1 - ac_t \ .$$

The Euler equation now becomes (after dividing over by a):

$$c_t = E_t\{c_{t+1}\} \ ,$$

which implies that

$$E_t \Delta c_{t+1} = 0 \ .$$

In words, the PIH model has the strong implication that the expected change in consumption is zero! (Read carefully, this does not mean that the change in consumption is zero, it means that the change has mean 0. Sometimes, people will say that the predicted change in consumption is zero, but the model does not predict no change in consumption only a mean-0 change.) Econometricians have spent much time testing this prediction. In order to test the model, we need to assume that the relation holds each period, in other words, that at each period t,

$$E_t \Delta c_{t+1} = 0 \ , \forall t \, .$$

In statistics, a variable which at any period t has period $t+1$ expectation equal to the period t value is called a *martingale*, so Hall's model predicts that consumption is a martingale (although it is often, imprecisely, referred to as a random walk).

Hall tested the model on aggregate data (it has many times and often with more success been tested on micro-data, but we will restrict ourself to macro here) and found only weak evidence against it. Later papers have often found more evidence against the result and we will shortly cover some of this material.