ORDINARY DIFFERENTIAL EQUATIONS PRELIMINARY EXAM.

THERE ARE 6 QUESTIONS. ATTEMPT 6 OUT OF 6 QUESTIONS.

SHOW ALL WORKING.

THIS IS A THREE HOUR CLOSED BOOK EXAM.

EACH QUESTION IS WORTH 20 POINTS.

GOOD LUCK.

(1) Let A be an $n \times n$ matrix. Let λ be a real eigenvalue of A and define $E(\lambda, k) = \{v \in \mathbf{R}^n : (A - \lambda I)^k v = 0\}$. Suppose that B is an $n \times n$ matrix that commutes with A i.e. $AB = BA$.

Show that $E(\lambda, k)$ is invariant under the flow generated by

$$\dot{x} = Bx, \qquad x(0) = x_0$$

i.e. if $x_0 \in E(\lambda, k)$ then $x(t) \in E(\lambda, k)$.

(b) Give an example of a 3×3 real matrix A such that the solution to

$$\dot{x} = A \qquad x(0) = x_0$$

is

(i) periodic with period 2π if x_0 lies in the plane spanned by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

(ii) limit to the origin if

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(2) (a) Suppose A is an $n \times n$ matrix and $B(t)$ is a continuous map, $(B : \mathbf{R} \to \mathbf{R}^n)$. Prove that all solutions of

$$\dot{x} = Ax + B(t)$$

are of the form

$$x(t) = e^{At}[\int_0^t e^{-As} B(s) ds + C]$$

where C is a constant vector in $\mathbf{R}^n$.

(b) Using part (a) or otherwise show that if $n = 2$ then all solutions to

$$\dot{x} = Ax + B(t), \qquad x(0) = x_0$$

limit to the origin if

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}$$

and the continuous map $B(t)$ satisfies $\lim_{t \to \infty} B(t) = 0$.

(3) (a) Give an example of a one-dimensional ordinary differential equation of form

$$\dot{x} = f(x), \qquad x(0) = x_0$$

where $f : \mathbf{R} \to \mathbf{R}$ is continuous but there exists more than one differentiable solution. (Make sure to prove your assertions).

(b) Is it true that a one-dimensional ordinary differential equation of form

$$\dot{x} = f(x), \qquad x(0) = x_0$$

where $f(x)$ is differentiable in x for all x has a differential solution $x(t)$ defined for all $t \geq 0$? If true briefly give a reason, if false give a counterexample.

(c) Consider the two-dimensional system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y(4 - 2x^2 - 3y^2) \end{aligned}$$

Show that this system has a non-trivial periodic solution.

Hint: Consider $\frac{dr}{dt}$ *where* $r = x^2 + y^2$ *and use Poincaré-Bendixson.*

(4) (a) Show that the equilibrium of the system

$$\begin{aligned}\dot{x} &= -2x + y^2 x \\ \dot{y} &= -3yx^2 + y^2 x^3\end{aligned}$$

at the origin is asymptotically stable by using a Lyapunov function (or otherwise).

(b) Consider the system

$$\begin{aligned}\dot{x} &= x + 2z^3 \\ \dot{y} &= -y + z^3 \\ \dot{z} &= -z\end{aligned}$$

(i) Find the stable E^s and unstable E^u subspaces for the equilibrium at the origin.

(ii) By solving the system explicitly find a formula for the stable manifold $W^s(0)$ and unstable manifold $W^u(0)$.

Hint: Solve iteratively starting with $\dot{z} = -z$ and use the result of 2(a).

(5) (a) Suppose

$$\dot{x} = f(x)$$

where $f(x, y) = (f_1(x, y), f_2(x, y))$ is a C^2 vector-valued function $f : \mathbf{R}^2 \to \mathbf{R}^2$ which satisfies $f_1(-x, y) = -f_1(x, y)$ for all (x, y). Show that the system has no non-trivial periodic solution containing the origin in its interior.

(b) Prove Gronwall's inequality:

If $g(t)$ is a continuous real-valued function that satisfies $g(t) \geq 0$ and

$$g(t) \leq C + K \int_0^t g(s)ds$$

for all $t \in [0, a]$ for positive constants $C, K > 0$ the for all $t > 0$

$$g(t) \leq Ce^{Kt}$$

(6) (a) Consider the two-dimensional system

$$\begin{aligned}\dot{x} &= y - x(x^2 + y^2 - \lambda)\\ \dot{y} &= -x - y(x^2 + y^2 - \lambda)\end{aligned}$$

where λ is a real parameter. Sketch phase portraits in the plane for $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ and describe the associated flows in these three settings.

Hint: it may help to change to polar coordinates.

(b) We say that two flows $\phi_t : U \to U$, $\psi_t : V \to V$ where U,V are open sets in R^n are C^1-conjugate if there exists a C^1 diffeomorphism $h : U \to V$ such that $h \circ \phi_t = \psi_t \circ h$ for all $t \in \mathbf{R}$ i.e. $h(\phi_t(x)) = \psi_t(h(x))$ for all $x \in U$. We say the flows are topologically conjugate if h is a homeomorphism.

(i) Suppose ϕ_t and ψ_t are topologically -conjugate. Show that equilibria and periodic orbits of ϕ_t are mapped to equilibria and periodic orbits of ψ_t by h. Need the period of corresponding periodic orbits be the same? Either prove or give a counterexample.

(ii) Suppose ϕ_t and ψ_t are C^1 -conjugate Show that if x_0 is an equilibrium point for ϕ_t and $h(x_0)$ the corresponding fixed point for ψ_t then the linearization of ϕ_t at x_0 has the same eigenvalues as the linearization of ψ_t at $h(x_0)$.