

Sample ODE Qualifying Exam Spring 2005

Be prepared to give statements of any definitions and/or theorems associated with the topics listed on the syllabus for the ODE PhD Written Exam.

1. Let A be a real $n \times n$ matrix, and suppose $\operatorname{re}(\lambda) < 0$ for all $\lambda \in \sigma(A)$. Suppose $f \in C([0, \infty), \mathbb{R}^n)$ and there is a number $M > 0$ so that $\|f(t)\| \leq M$ for all $t \geq 0$. Prove there is a number $K > 0$ so that

$$\limsup_{t \rightarrow \infty} \|u(t)\| \leq K$$

for all $u \in C^1([0, \infty), \mathbb{R}^n)$ satisfying

$$u'(t) = Au(t) + f(t)$$

2. Let $f \in C^1(\mathbb{R}, \mathbb{R})$ and let $u, v \in C^1([0, \infty), \mathbb{R})$ such that

$$u'(t) \leq f(u(t))$$

$$v'(t) \geq f(v(t))$$

and $u(0) \leq v(0)$. Prove that $u(t) \leq v(t)$ for all t .

3. Let $M > 0$. Suppose $f \in C^1(\mathbb{R}, \mathbb{R})$ and $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that if $u, v \in C^1([0, \infty), \mathbb{R})$ satisfy

$$\begin{aligned} u'(t) &= f(u(t)), & t \geq 0 \\ v'(t) &= f(v(t)), & t \geq 0 \end{aligned}$$

then

$$|u(t) - v(t)| \leq |u(0) - v(0)| \exp(Mt) \text{ for all } t \geq 0$$

4. Let a, u_0, v_0, w_0 be positive values. The initial value problem given by

$$\begin{cases} u'(t) = w(t) - au(t)v(t), & t \geq 0 \\ v'(t) = w(t) - au(t)v(t), & t \geq 0 \\ w'(t) = au(t)v(t) - w(t), & t \geq 0 \\ u(0) = u_0, v(0) = v_0, w(0) = w_0 \end{cases} \quad (1)$$

can be used to analyze the behavior of chemical concentrations $[A]$, $[B]$ and $[C]$ for the hypothetical reaction



- (a) Show that this system has a unique maximal solution, and that $u(t)$, $v(t)$ and $w(t)$ are nonnegative on the maximum interval of existence.
- (b) Show that $u(t) + v(t) + 2w(t) = u_0 + v_0 + 2w_0$, and explain why this implies the solution to (1) exists globally.
- (c) Determine the asymptotic behavior of each of $u(t)$, $v(t)$ and $w(t)$ as $t \rightarrow \infty$.

5. The two component Brusselator is given by

$$\begin{cases} u'(t) = A + u(t)^2v(t) - (B + 1)u(t), & t \geq 0 \\ v'(t) = Bu(t) - u(t)^2v(t), & t \geq 0 \\ u(0) = u_0, v(0) = v_0 \end{cases} \quad (2)$$

where A, B, u_0 and v_0 are positive constants.

- (a) Show that this system has a unique maximal solution, and that $u(t)$ and $v(t)$ are positive on their interval of existence.
- (b) Show there exist constants $L, K > 0$ so that

$$(u(t) + v(t))' \leq L - K(u(t) + v(t))$$

and explain why this implies solutions to (2) are global.

- (c) Explain how the results above can be combined with the Poincare Bendixson theorem to prove that there exists a nontrivial asymptotically stable time periodic solution to (2) whenever $B > 1 + A^2$.

6. The general Lotka-Volterra model is given by

$$\begin{cases} P_i'(t) = \left(b_i - \sum_{j=1}^n a_{i,j}P_j(t) \right) P_i(t), & t \geq 0, i \in \{1, \dots, n\} \\ P_i(0) = Q_i \end{cases} \quad (3)$$

where $b = (b_i) \in \mathbb{R}^n$, $A = (a_{i,j}) \in \mathbb{R}_{n,n}$ and $Q = (Q_i) \in \mathbb{R}_+^n$. Show that this system has a unique maximal solution, and that $P(t) \in \mathbb{R}_+^n$ on its interval of existence. Show (by example) that it is possible for solutions to (3) to not exist globally. Finally, show that if the matrix A is positive definite then solutions to (3) are bounded and exist globally.

7. The Lorenz system is given by

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= \rho x - y - xz \\ z' &= xy - \beta z \end{aligned}$$

with $\sigma, \rho, \beta > 0$. Show that the only steady state associated with this system is $(0, 0, 0)$. Then give a graph in the positive quadrant of the $\beta\rho$

plane showing the regions of asymptotic stability and instability for the steady state $(0, 0, 0)$ that can be determined from linearized stability. In each of these regions, indicate the dimensions of the stable and unstable manifolds, and determine appropriately dimensioned subspaces that will be tangent to these manifolds at $(0, 0, 0)$.

8. Suppose $F \in C^2(\mathbb{R}^n, \mathbb{R})$ and

$$\lim_{\|u\| \rightarrow \infty} F(u) = \infty$$

Show that if $u_0 \in \mathbb{R}^n$ then there is a unique global solution to the system

$$\begin{aligned} u'(t) &= -\nabla F(u(t)), \quad t \geq 0 \\ u(0) &= u_0 \end{aligned}$$

In addition, determine the asymptotic behavior in the case when F is strictly convex.

9. Suppose $f \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^n)$ such that f is locally Lipschitz with respect to its second argument; i.e. there is a function $g \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k, \mathbb{R})$ such that

$$\|f(t, u, \alpha) - f(t, v, \alpha)\| \leq g(t, u, v, \alpha) \|u - v\|$$

whenever $(t, u, v, \alpha) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k$. Suppose $t_0 \in \mathfrak{R}$, $u_0 \in \mathfrak{R}^n$ and $\alpha \in \mathfrak{R}^k$, and let u be the unique maximal solution to

$$\begin{cases} u'(t) = f(t, u(t), \alpha) \\ u(t_0) = u_0 \end{cases}$$

on an interval I . Assume $a < t_0 < b$ such that $[a, b] \subset I$. Show that if $\varepsilon > 0$ then there is a value $\delta > 0$ so that if $v_0 \in \mathfrak{R}^n$ and $\beta \in \mathfrak{R}^k$ with

$$\|u_0 - v_0\| + \|\alpha - \beta\| < \delta$$

then the interval of existence of the maximal solution v to

$$\begin{cases} v'(t) = f(t, v(t), \beta) \\ v(t_0) = v_0 \end{cases}$$

contains $[a, b]$ (i.e. v exists as a solution on $[a, b]$) and

$$\|u(t) - v(t)\| < \varepsilon \text{ for all } a \leq t \leq b.$$

10. Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and suppose $T > 0$ such that there is a nontrivial time periodic solution v to the system

$$v'(t) = f(v(t))$$

with period T . Define $\tilde{u} = v(0)$ and let $\varepsilon > 0$.

- (a) Define the notions of (local) stability, asymptotic stability, and instability for the time periodic solution v .
- (b) Explain why $f(\tilde{u}) \neq 0$.
- (c) Let V be the affine space given by $V = \{\tilde{u} + v \mid v \cdot f(\tilde{u}) = 0\}$. Show there are relatively open neighborhoods U_ε and W_ε of \tilde{u} in V so that $W_\varepsilon \subseteq U_\varepsilon$ if $u_0 \in W_\varepsilon$ and $u(t; u_0)$ solves

$$\begin{aligned} u' &= f(u) \\ u(0) &= u_0 \end{aligned}$$

then there exists a value $\tau(u_0) > 0$ such that $u(t; u_0) \notin U_\varepsilon$ for $0 < t < \tau(u_0)$, $u(\tau(u_0); u_0) \in U_\varepsilon$, and $|T - \tau(u_0)| < \varepsilon$.

- (d) Define the first return map $G : W_\varepsilon \rightarrow U_\varepsilon$ via $G(u_0) = u(\tau(u_0); u_0)$.
- (e) Let $\Phi(t)$ solve

$$\begin{aligned} \Phi'(t) &= f'(u(t; \tilde{u})) \Phi(t) \\ \Phi(0) &= I_{n \times n} \end{aligned}$$

How is the spectrum of $G'(\tilde{u})$ related to the spectrum of $\Phi(T)$?

- (f) Give conditions on the spectrum of $\Phi(T)$ (and hence $G'(\tilde{u})$) that guarantees $u(t; \tilde{u})$ is an asymptotically stable time periodic solution.